Celestial Mechanics and Gravity

Learners' Space Astronomy

Contents

Introduction

Celestial Mechanics, the study of motion of celestial bodies was one of the main branches of astronomy up until the end of the 19th century. Primarily, it was meant to explain the motion of planets and satellites under the Sun.
It started with Ptolemy, who proposed the geocentric model of the universe and epicycles. The model assumed that Earth is at the center of the universe and all the celestial bodies revolved around the Earth. This model couldn't explain the 'retrograde' motion of the planets. So the concept of epicycles (planets have additional motion in smaller cycles) was introduced. Of course, all the above was trashed by later theories and observations. For example, Nicholas Copernicus proposed the heliocentric model. His proposition was far more consistent with all the observations and far simpler. Later in the 16th century, Tycho Brahe made very accurate observations before the invention of the telescope, which led his apprentice, Johannes Kepler, to derive three empirical laws about planetary motion.
We will also look into interesting kinds of stuff about gravity and the effects caused by it that will touch General Relativity in some way.

Two Body System

The two-body problem refers to the mathematical problem of predicting and describing the motion of two objects that gravitationally interact. It is a fundamental problem in classical mechanics and has applications in various fields, including celestial mechanics.
In the context of celestial mechanics, the two-body problem typically refers to the motion of two celestial bodies, such as a planet and a star, or a satellite and a planet. The gravitational force between the two bodies causes them to attract each other and influences their orbital motion.
Kepler's laws of planetary motion are a set of three empirical laws formulated by the German astronomer Johannes Kepler in the early 17th century, these laws describe the motion of the planet around the sun which is derived from extensive observational data

2.1 Kepler's Laws

Kepler's First Law A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse.

Kepler's Second Law A line connecting a planet to the Sun sweeps out equal areas in equal time intervals.
i.e.

$$\frac{dA}{dt} = \frac{l}{2\mu} \tag{2.1}$$

where l is the angular momentum and μ is the reduced mass
We will get back to this later when we will see Two Body Problem
Kepler's Third Law The Harmonic Law

$$P^2 \propto a^3 \tag{2.2}$$

where P is the time period and a is the semi-major axis of the ellipse

Trivia

Kepler published the first textbook of Copernican astronomy, *Epitome Astronomiae Copernicanae* (1618–21; *Epitome of Copernican Astronomy*). The Epitome began with the elements of astronomy but then gathered together all the arguments for Copernicus's theory and added to them Kepler's harmonics and new rules of planetary motion. This work would prove to be the most important theoretical resource for the Copernicans in the 17th century. Galileo and Descartes were probably influenced by it, and inspired Newton to come up with his theory of Gravitation.

2.2 Solving 2 body problem

All of the work done till this was empirical. Newton was the first to precisely define force, momentum, and all these quantities. He backtracked the observations of Kepler and noticed that an inverse square law for force would explain all of those observations.

Now we will show how to arrive at these results through Newton's laws of motion and gravity. Look at the diagram shown below. By Newton's law, we can write:

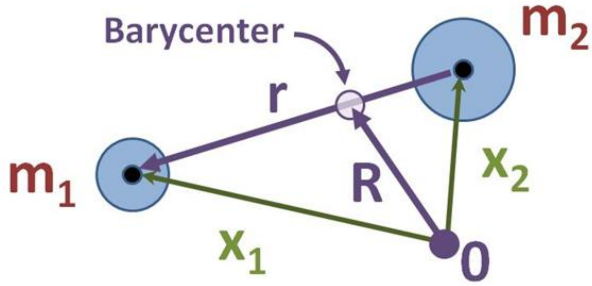


Figure 2.1: 2 body system

$$m_1 \frac{d^2 x_1}{dt^2} = \mathbf{F}_{12} \tag{2.3}$$

$$m_2 \frac{d^2 x_2}{dt^2} = \mathbf{F}_{21} \tag{2.4}$$

As, $\mathbf{r} = \mathbf{x_1} - \mathbf{x_2}$
and also we know that $\mathbf{F}_{21} = -\mathbf{F}_{12}$, we can write(by subtracting Eq.(2.4) from Eq.(2.3);

$$\frac{d^2 x_1}{dt^2} - \frac{d^2 x_2}{dt^2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_{12} \tag{2.5}$$

Simplifying this will lead to the below equation;

$$\mu \frac{d^2 r}{dt^2} = \mathbf{F}_{12} = \mathbf{F} \tag{2.6}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

This reduces the 2 body problem to a single body (Motion of a body under a central force. Also, we will be first solving this for the general central force, so that we can get a more general idea of the body's motion and only later we will be putting $\mathbf{f(r)} = -\frac{Gm_1m_2}{r^2}$). Taking the force as central force (as in the case of other planetary and celestial orbits),i.e.,

$$\mathbf{F} = \mathbf{f(r)}\hat{r}$$

Now observe the torque;

$$\tau = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{f(r)}\hat{r} \tag{2.7}$$

The vector $\mathbf{r}$ and the force are in the direction of r. So,

$$\tau = 0 \tag{2.8}$$

Now recall polar coordinates. We can write the acceleration of any particle as

$$a = [\ddot{r} - r\dot{\theta}^2]\hat{r} + [2\dot{r}\dot{\theta} + r\ddot{\theta}]\hat{\theta} \tag{2.9}$$

Multiplying the above acceleration with μ gives us the force.

Now the force is in the radial direction, the force in the direction of θ is zero, and hence, so is the acceleration. So we have,

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{2.10}$$

Multiplying the above eq. by r will clearly show us that LHS is the derivative of $r^2\frac{d\theta}{dt}$.
That means,

$$\mu r^2 \frac{d\theta}{dt} = C \tag{2.11}$$

where C is some constant.

Eq.(2.11) is the angular momentum. The fact that the angular momentum is constant can also be checked by Eq.(2.8). As the torque is zero, the angular momentum must be constant over time
Now we need to know the energy to find out the trajectory/path that the body will follow.
We know that the total energy(E) can be written as,

$$E = \frac{1}{2}\mu v^2 + U(r) \tag{2.12}$$

writing v in terms of polar coordinates,

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2$$

The total energy will finally become,

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) \tag{2.13}$$

Manipulate the equation and it will become a differential equation in r.

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E - U(r) - \frac{l^2}{2\mu r^2})} \tag{2.14}$$

We can write $\frac{\dot{r}}{\dot{\theta}} = \frac{dr}{d\theta}$
And $\dot{\theta}$ is nothing but equals to $\frac{l}{\mu r^2}$
Substituting this in Eq.(2.14) and integrating it on both sides, we get

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{\frac{l}{\mu r^2} dr}{\pm\sqrt{\frac{2}{\mu}(E - U(r) - \frac{l^2}{2\mu r^2})}} \tag{2.15}$$

Now, we put the value of force for a system that is bound by gravity, the force **f(r)** is equal to $-\frac{Gm_1m_2}{r^2}$.
To calculate potential energy($U(r)$), we integrate the force, and we get $U(r) = -\frac{Gm_1m_2}{r}$
So in the energy expression,

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \tag{2.16}$$

the first term determines the kinetic energy and the last two terms are the effective potential (effective because it combines multiple, perhaps opposing, effects into a single potential. It is the sum of the 'opposing' centrifugal potential energy with the normal potential of the system).

Solve the integral in Eq.(2.15) by taking $U(r)$ as $-\frac{Gm_1m_2}{r}$ and taking $\frac{l}{r}$ or just $\frac{1}{r}$ as u, and it will become a standard integral of form $\frac{dx}{\sqrt{ax^2+bx+c}}$, which is easily solvable and give the integral of form $-\frac{1}{\sqrt{-a}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}}$
After solving the integral by putting r_0 as $\frac{l^2}{\mu Gm_1m_2}$ and θ_0 as $-\frac{\pi}{2}$, we get ,

$$r = \frac{r_0}{1 - \epsilon\cos\theta} \tag{2.17}$$

where $\epsilon = \sqrt{1 + \frac{2El^2}{\mu(Gm_1m_2)^2}}$.
Eq.(2.17) represents the equation of a conic section in polar coordinates, where r_0 equals semi-latus rectum and ϵ is the eccentricity of the conic and θ is the angle from the line joining a focus and its perigee(also known as a true anomaly).
Manipulating it and putting $r\cos\theta$ as x and r as $\sqrt{x^2+y^2}$, we get

$$(1-\epsilon^2)x^2 - 2\epsilon r_0 x + y^2 = r_0^2 \tag{2.18}$$

Now there are 4 cases:

- $\epsilon > 1$, which implies $E > 0$ and Eq.(2.18) becomes a sort of an equation of Hyperbola.
- $\epsilon = 1$, which implies $E = 0$ and Eq.(2.18) becomes equation of a Parabola.
- $0 < \epsilon < 1$, which implies $E < 0$ and Eq.(2.18) becomes equation of an Ellipse
- $\epsilon = 0$, which gives a special case of Circular orbit

This clearly shows the condition of a bounded orbit (of course parabola and hyperbola are not bound just by looking at their shapes), which is that the energy should be $-ve$.
Hence the orbit of a planet/star bound to another star is elliptical.
Now we will try to find the total energy of the 2 body systems. We know that $r_{max} + r_{min} = 2a$

(perigee and apogee are the points for min and max resp.). and these min and max values of r occur at $\theta = 0$ and π.
Substituting these values of θ in the above equation gives us,

$$a = \frac{r_0}{1-\epsilon^2} \tag{2.19}$$

Substitute value of r_0 and ϵ as $\frac{l^2}{\mu G m_1 m_2}$ and $\sqrt{1+\frac{2El^2}{\mu(Gm_1m_2)^2}}$ resp. we get,

$$E = -\frac{Gm_1m_2}{2a} \tag{2.20}$$

Moving forward to Prove Kepler's 2nd law
Now look at the below equation,

$$\frac{l}{\mu} = r^2\frac{d\theta}{dt} \tag{2.21}$$

The above equation has a similarity with the equation $v = r\frac{d\theta}{dt}$. The only difference is one higher power of r, which can be interpreted as surface velocity instead of linear velocity, and surface velocity is $C\times(\textit{time derivative of area vector})$, same as linear velocity is time derivative of the radius vector. So sometimes it's also referred to as Areal velocity. So,

$$\frac{dA}{dt} = C\frac{l}{\mu} = Cr^2\frac{d\theta}{dt} \tag{2.22}$$

Now taking a special case of Circular orbit(where r is constant), so we can write an elemental Area as,

$$dA = \frac{1}{2}r^2\theta$$

Taking derivative w.r.t time , we get $C = \frac{1}{2}$.
From here the value of C is equal to 0.5.
Hence we proved Kepler's 2nd law also.

Integrate the above surface velocity equation,

$$\int_{ellipse} dA = \frac{l}{2\mu}\int dt \tag{2.23}$$

Integrating it over the whole time period of one revolution of the planet gives,

$$\pi ab = \frac{l}{2\mu}T \tag{2.24}$$

As in Eq.(2.18), r_0 = semi-latus rectum , so
$r_0 = \frac{b^2}{2a} = \frac{l^2}{\mu G m_1 m_2}$.
Substitute l from this in Eq.(2.24) and square on both sides. You will get,

$$T^2 = \frac{4\pi^2}{G(m_1+m_2)}a^3 \tag{2.25}$$

This gives us a brief idea of how two body system works.

Trivia

You can go to this link to visualize how a two-body system works by varying masses of the bodies. You will find it interesting(especially the Sun-Earth-Moon system)

Three Body system

3.1 Introduction

We just learned about the "two-body problem" which has been solved. However, adding another body makes things much more complicated. The typical three-body problem involves 18 first-order differential equations(see further reading), which through the use of calculus and conservation equations can be reduced to 6 and has not yet been solved. However, we can look at a more restricted case to get a general idea.

The restricted three-body problem has two large masses orbiting at their common center of mass. A third relatively smaller body is then introduced.

We will look at a case where all the bodies lie on the X-Y plane. There are two large bodies with mass m_1 and m_2 with their center of mass at the origin, which are moving around it in a circle, and a third small body with a mass m_3 being p_1 and p_2 distant from the two bodies respectively.

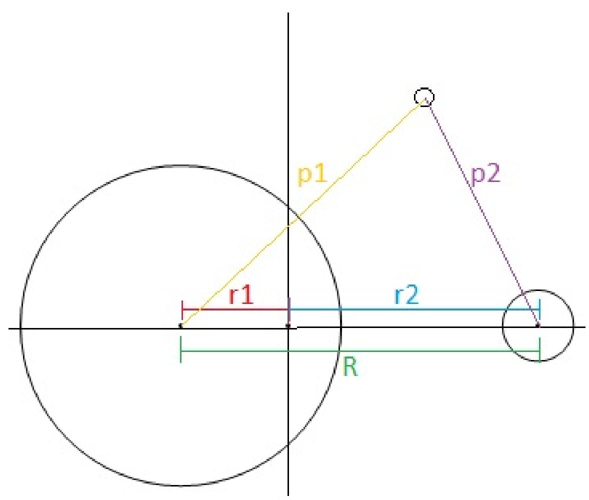


Figure 3.1: 3-Body Visualization

Without loss of generality, let us assume $m_1 \geq m_2$ and $R = 1$. We define $\mu = \frac{m_2}{m_1+m_2}$. Therefore, $r_1 = -\mu$ and $r_2 = 1 - \mu$.

Now, we determine the equations of motion of the third body. The kinetic energy is represented by

$$T = \frac{1}{2}m_3(\dot{x}^2 + \dot{y}^2)$$

and the potential energy by:

$$V = -\frac{Gm_1m_3}{p_1} - \frac{Gm_2m_3}{p_2}$$

where

$$p_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2}, \quad p_2 = \sqrt{(x - x_2)^2 + (y - y_2)^2}$$

Now, we will shift to a rotational frame of reference, centered at the origin moving with the same angular velocity as the first two bodies.

$$\omega = \sqrt{\frac{G(m_1 + m_2)}{(r_1 + r_2)^3}} = \sqrt{G(m_1 + m_2)} \tag{3.1}$$

The co-ordinates change as:

$$x'(t) = x\cos\omega t - y\sin\omega t$$

$$y'(t) = y\cos\omega t + x\sin\omega t$$

. New kinetic energy will be:

$$T = \frac{1}{2}m_3(\dot{x}^2 + \dot{y}^2 + 2x\omega\dot{y} - 2y\omega\dot{x} + \omega^2(x^2 + y^2))$$

The equations of motion, taking into account the Coriolis force and the centrifugal force is:

$$m_3\ddot{x} - 2m_3\omega\dot{y} - m_3x\omega^2 = -\frac{Gm_1m_3(x - x_1)}{p_1^3} - \frac{Gm_2m_3(x - x_2)}{p_2^3} \tag{3.2}$$

$$m_3\ddot{y} + 2m_3\omega\dot{x} - m_3y\omega^2 = -\frac{Gm_1m_3(y - y_1)}{p_1^3} - \frac{Gm_2m_3(y - y_2)}{p_2^3} \tag{3.3}$$

To these equations we put the values of $G = \frac{\omega^2}{m_1+m_2}$ and then $\mu = \frac{m_2}{m_1+m_2}$. Cancelling m_3, we get:

$$\frac{d^2x}{dt^2} = 2\omega\frac{dy}{dt} + x\omega^2 - \frac{\omega^2(1-\mu)(x - x_1)}{p_1^3} - \frac{\omega^2\mu(x - x_2)}{p_2^3} \tag{3.4}$$

$$\frac{d^2y}{dt^2} = -2\omega\frac{dx}{dt} + y\omega^2 - \frac{\omega^2(1-\mu)(y - y_1)}{p_1^3} - \frac{\omega^2\mu(y - y_2)}{p_2^3} \tag{3.5}$$

Next, we rewrite the time variable $t = \frac{\tau}{\omega}$ to scale out all the ω terms. After cancelling ω, we finally get:

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x - x_1)}{p_1^3} - \frac{\mu(x - x_2)}{p_2^3} \tag{3.6}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)(y - y_1)}{p_1^3} - \frac{\mu(y - y_2)}{p_2^3} \tag{3.7}$$

Since we are using a rotational co=ordinate system, without loss of generality, we can say that $(x_1, y_1) = (-\mu, 0)$ and $(x_2, y_2) = (1 - \mu, 0)$. Putting these into the above equation, the final equations we get are:

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x + \mu)}{p_1^3} - \frac{\mu(x - 1 + \mu)}{p_2^3} \tag{3.8}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{p_1^3} - \frac{\mu y}{p_2^3} \tag{3.9}$$

with

$$p_1 = \sqrt{(x+\mu)^2 + y^2}, \quad p_2 = \sqrt{(x-1+\mu)^2 + y^2}$$

3.2 Lagrange Points

Euler and Lagrange proved that there are 5 equilibrium points(i.e., they appear to be at rest concerning the reference frame associated wrt the first two bodies) for the third body. These points are called Lagrange points. Since these points are at rest in the rotational frame, the acceleration and velocity are put to zero to get:

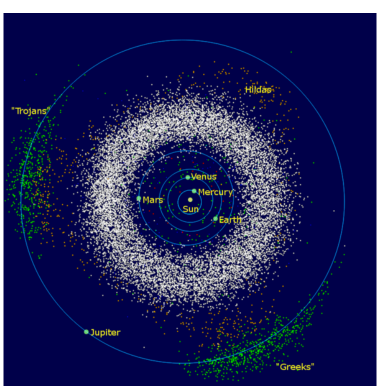
$$x = \frac{(1-\mu)(x+\mu)}{p_1^3} + \frac{\mu(x-1+\mu)}{p_2^3} \tag{3.10}$$

$$y = \frac{(1-\mu)y}{p_1^3} + \frac{\mu y}{p_2^3} \tag{3.11}$$

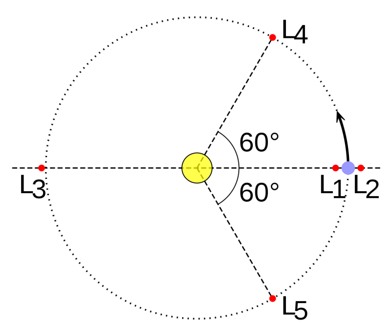
Euler discovered three points, which were collinear with the two bodies. Thus, by putting $y = 0$,

$$x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x-1+\mu)}{|x-1+\mu|^3} \tag{3.12}$$

This equation has 3 roots and these points are labeled as L_1, L_2, and L_3. In most cases, μ is small, and L_1 and L_2 lie on either side of the smaller body, and L_3 lies on the far side of the larger body. Later, Lagrange discovered the location of the other two points to be at equilateral triangles from the two large bodies. Here, $p_1 = p_2 = 1$, $x = \mu - 1/2$ and $y = \pm\frac{\sqrt{3}}{2}$. They are called L_4 and L_5. It is to be noted that the first 3 Lagrange points are unstable equilibriums, and L_4 and L_5 are stable if $\mu < \frac{1}{2}(1 - \sqrt{\frac{23}{27}}) = 0.0385208965$. Almost all values of μ in our Solar System is less than this value. Trojan is the best example for stability at points L_4 and L_5. In the Solar System, most known Trojans share the orbit of Jupiter. In other planetary orbits only nine Mars trojans, 28 Neptune trojans, two Uranus trojans, and two Earth trojans, have been found to date. A temporary Venus trojan is also known.



(a) Example for stability at L_4 and L_5



(b) Lagrange points

Figure 3.2

Trivia

Some bodies present at different Lagrange points of different systems are:

- The James Webb Space Telescope, a powerful space observatory, is located at L_2 of the Sun-Earth System. This location protects the telescope from the light and heat from the Sun.
- An artificial satellite called the Deep Space Climate Observatory (DSCOVR) is located at L_1 to study solar wind coming toward Earth from the Sun and to monitor Earth's climate, by taking images and sending them back.
- The Sun–Earth L_4 and L_5 points contain interplanetary dust and at least two asteroids.
- The Earth–Moon L_4 and L_5 points contain concentrations of interplanetary dust, known as Kordylewski clouds.
- The Sun–Neptune L_4 and L_5 points contain several dozen known objects, the Neptune trojans.
- In binary stars, the Roche lobe has its apex located at L_1; if one of the stars expands past its Roche lobe, then it will lose matter to its companion star, known as Roche lobe overflow.
- One version of the giant impact hypothesis postulates that an object named Theia formed at the Sun–Earth L_4 or L_5 point and crashed into Earth after its orbit destabilized, forming the Moon.

3.3 Some nice examples of 3 body system

Michel Hénon and Roger A. Broucke each found a set of solutions that form part of the same family of solutions: the Broucke–Henon–Hadjidemetriou family. In this family, the three objects all have the same mass and can exhibit both retrograde and direct forms. In some of Broucke's solutions, two of the bodies follow the same path

In 1993, a zero angular momentum solution with three equal masses moving around a figure-eight shape was discovered numerically by Cris Moore

Finally, some 20 periodic solutions of the 3 body problem were discovered, as shown below:

Refer to the links given below each image to understand it with the help of a GIF.

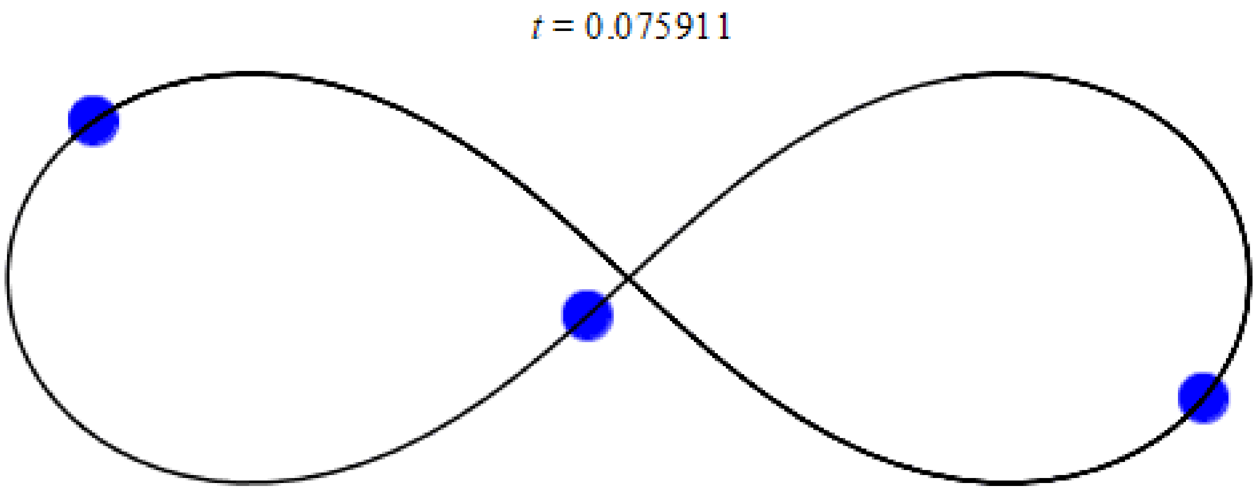


Figure 3.3: https://en.wikipedia.org/wiki/File:Three_body_problem_figure-8_orbit_animation.gif

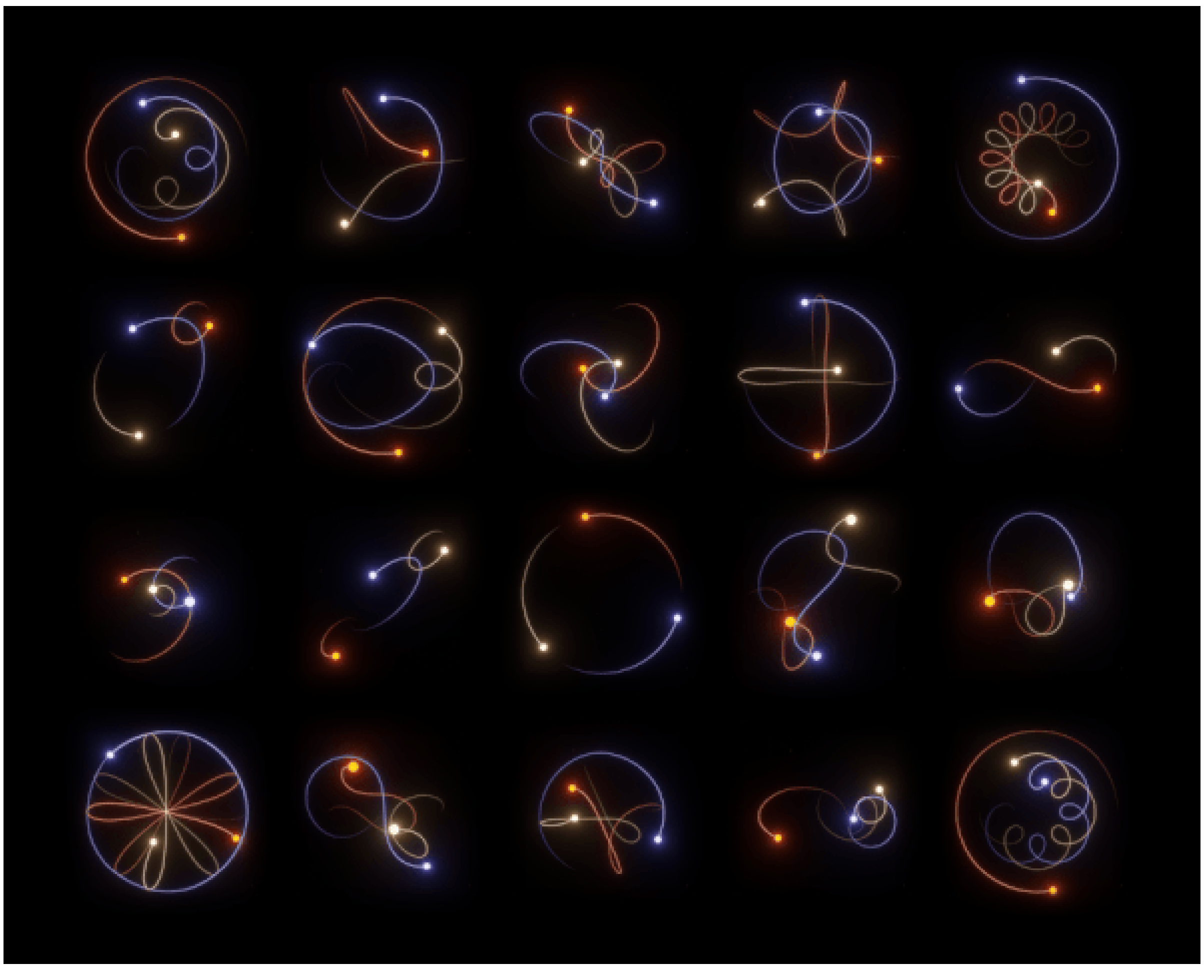


Figure 3.4: https://en.wikipedia.org/wiki/File:5_4_800_36_downscaled.gif

3.4 Approaching the n-body problem

As the heading suggests, we will be attempting to find certain properties of a system of n-bodies that are gravitationally bound. First, we write the equations of motion for all the bodies. Denoting all their positions by $\vec{r_i}$ and velocities with $\vec{v_i}$ for $i = 0, \ldots, n-1$ and writing all the differential equations:

$$\frac{d\vec{r_i}}{dt} = \vec{v_i}, \quad \frac{d\vec{v_i}}{dt} = \sum_{i \neq j} \frac{Gm_i m_j (\vec{r_j} - \vec{r_i})}{|r_j - r_i|^3}, \qquad i = 0, 1, \ldots, n-1$$

Notice that there are a total of $6n$ first-order differential equations for bodies in three dimensions. If we restrict them to two dimensions, combine the velocity vector equations, and use the help of conservation quantities: energy, linear momentum, and angular momentum, we are still left with too many equations. We can use numerical integration to compute their trajectories given their initial position and velocities.

Henceforth, it is possible to derive statistical results for such a system. One such result is the **Virial theorem** which states that in a gravitationally bounded system, the time average of the total kinetic energy is negative half of the time average of the total potential energy of that system over a long time. i.e.,

$$\langle T \rangle = -\frac{1}{2} \langle U \rangle$$

This only assumes that the $\vec{r_i}$s and $\vec{v_i}$s remain bounded over time.

Trivia

The viral theorem is an important statistical result that can help us in calculating the average kinetic energy of the system even in very complicated systems. Apart from classical mechanics, it has been used in thermodynamics, quantum mechanic,s and even relativistic systems! (See: wiki)

Satellites

4.1 Introduction

Since October 4th, 1957, many hundreds of artificial satellites have been placed in orbit around the Earth. We have seen that Newton himself showed that if the projectile was given a sufficient velocity outside the Earth's atmosphere, it would become a satellite of the Earth. But it was only by the development of the rocket during and after the Second World War that a means was provided of imparting to a payload of instruments the velocity necessary to keep it in orbit. Artificial satellites are subject to Newton's laws of motion and the law of gravitation. They usually obey Kepler's laws very closely. If the Earth were a point-mass and no other force acted upon the satellite, a satellite would obey Kepler's laws exactly and remain in orbit forever. Many forces, however, may act on the satellite. Among these forces are:

(1) the Earth's gravitational field,

(2) the gravitational fields of the Sun, Moon, and the planets,

(3) the Earth's atmosphere and

(4) the Sun's radiation pressure.

In almost every case, the orbital changes produced by the Sun, Moon, and the planets are so small that they can be neglected. Only in the case of those artificial satellite orbits that take the satellite many thousands of kilometers away from the Earth does the disturbing effect of the Moon have to be considered. Even then, it is still small. As a result of the momentum associated with photons within any flow of radiation, the flux produces a pressure or force on any surface which intercepts the radiation. For a satellite whose size is large (for example a balloon satellite) and whose mass is small, the Sun's radiation pressure can produce large changes in the satellite's orbit over many months. For all other satellites, the orbital changes due to solar radiation pressure are negligible unless orbital positions are required to very high precision. The two main causes of change in a satellite orbit are, therefore, the departure of the Earth's shape from that of a perfect sphere and the drag due to the Earth's atmosphere on those satellites being low enough to experience it

4.2 Transfer of Satellites

In the below figure, we have two circular, coplanar orbits of radii a_1 and a_2 astronomical units (AU).

Remember from earlier, that the energy per unit mass of each orbit can be written as:

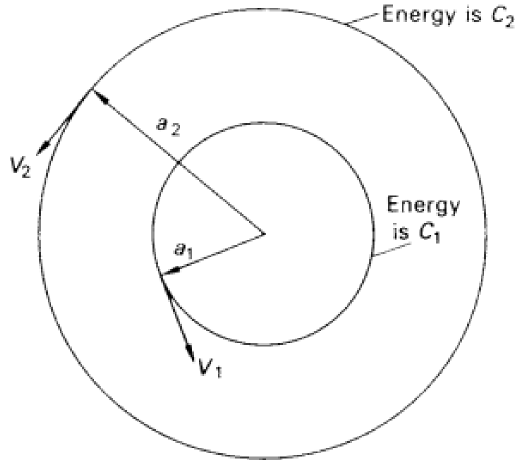


Figure 4.1: Description of two co-planar orbits

$$C_1 = \frac{1}{2}V^2 - \frac{GM}{a_1}$$

The reason for using C_1 instead of standard E_1 is because we are writing an expression of Energy per unit mass of the revolving body, and not just Energy, So to avoid confusion I've taken it as C_1.
We get,

$$C_1 = -\frac{GM}{2a_1}$$

Similarly,

$$C_2 = -\frac{GM}{2a_2}$$

Now $a_2 > a_1$, so that $C_1 < C_2$, in other words, a change of orbit would be a change of energy. This change of energy is brought about by the rocket engine. By imparting a velocity increment ΔV to the rocket, it changes its kinetic energy and, hence, its total energy. Hohmann studied how this could be most effectively done.
If the increment ΔV is applied along the instantaneous velocity vector $\mathbf{V}$, then the maximum increase in kinetic energy is achieved for a given burn, i.e. the full effect of ΔV is added to $\mathbf{V}$. If it is desired to decrease the kinetic energy, the velocity increment ΔV would be applied in the opposite direction to $\mathbf{V}$.

Hohmann showed that, in practice, the most economical transfer orbit between circular, coplanar orbit was an elliptical orbit cotangential to inner and outer orbits at perihelion and aphelion respectively. It is shown in the above figure as ellipse *APB*. Only one-half of the transfer orbit is used. At A, the rocket engine is fired to produce a velocity increment ΔV_A, applied tangentially to place the vehicle in the transfer orbit. The vehicle coasts round the half-ellipse *APB*, reaching aphelion at B. If no further change in energy took place, the vehicle would coast onwards along the mirror half of textitAPB to return eventually to A. A second impulse is, therefore, required

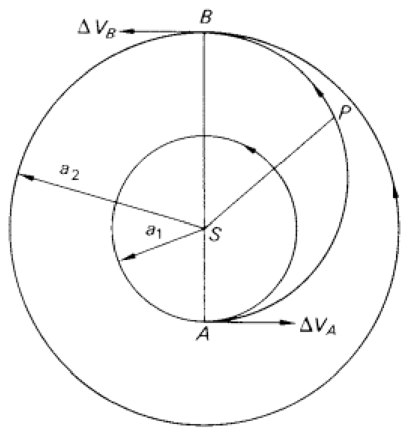


Figure 4.2: A Hohmann transfer orbit APB between two circular, co-planar orbits

to produce a second velocity increment ΔV_B. This is again obtained by firing the rocket engine tangentially and the rocket enters the outer circular orbit of radius a_2 AU.
Such transfer orbits are known as **Hohmann least-energy two-impulse cotangential orbits**.
By solving for this elliptical orbit, we can easily calculate the increment in velocity at points A and B.
Which comes out to be:

$$\Delta V_A = \sqrt{\frac{GM}{a_1}}\left[\sqrt{\frac{2a_2}{a_1+a_2}} - 1\right]$$

$$\Delta V_B = \sqrt{\frac{GM}{a_2}}\left[1 - \sqrt{\frac{2a_1}{a_1+a_2}}\right]$$

Trivia

There are other viable methods for orbit transfers as well:

- Bi-elliptic transfer
- Low thrust relative orbit transfer

4.3 Gravitational Slingshot

A gravity assist, gravity assist maneuver, swing-by, or generally a gravitational slingshot in orbital mechanics, is a type of spaceflight flyby which makes use of the relative movement (e.g. orbit around the Sun) and gravity of a planet or other astronomical object to alter the path and speed of a spacecraft, typically to save propellant and reduce expense.

Gravity assistance can be used to accelerate a spacecraft, that is, to increase or decrease its speed or redirect its path. The "assist" is provided by the motion of the gravitating body as it pulls on the spacecraft. Any gain or loss of kinetic energy and linear momentum by a passing spacecraft is correspondingly lost or gained by the gravitational body, by Newton's Third Law. The gravity assist maneuver was first used in 1959 when the Soviet probe Luna 3 photographed the far side of Earth's Moon and it was used by interplanetary probes from Mariner 10 onward, including the two Voyager probes' notable flybys of Jupiter and Saturn.

You can look at these links for the animations given below for visualization:

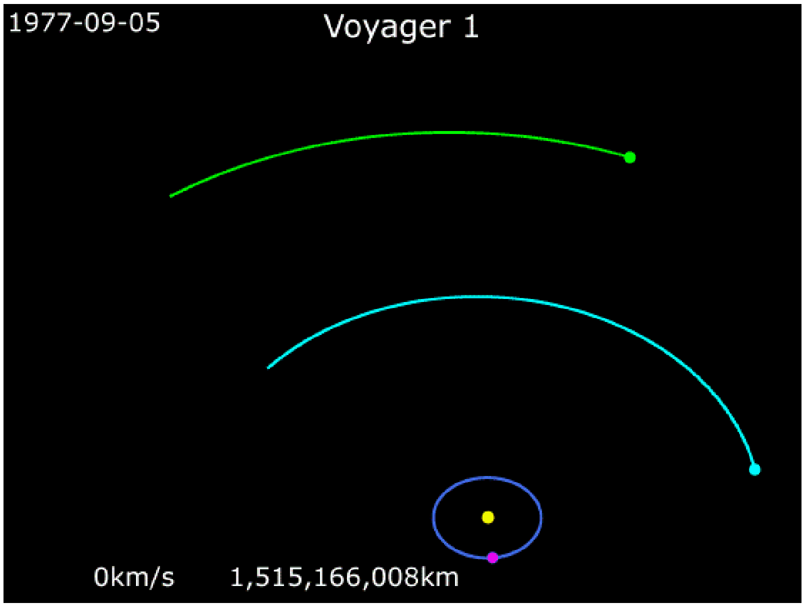


Figure 4.3: https://en.wikipedia.org/wiki/File:Animation_of_Voyager_1_trajectory.gif

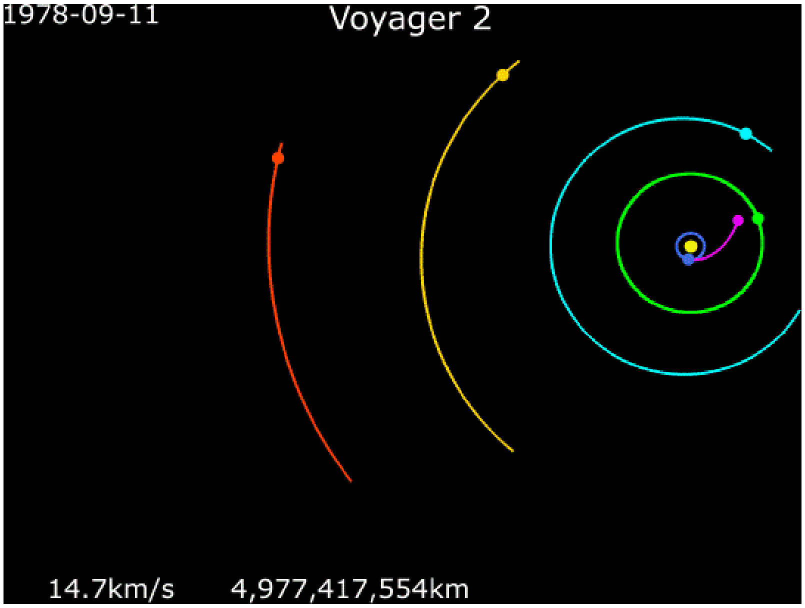


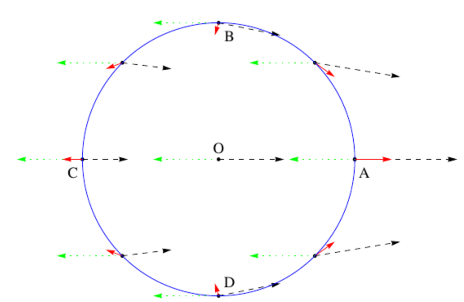
Figure 4.4: https://en.wikipedia.org/wiki/File:Animation_of_Voyager_2_trajectory.gif

Tides

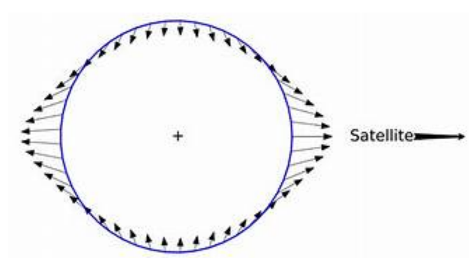
The regular rise and fall of the ocean's waters are known as tides. Along coasts, the water slowly rises over the shore and then slowly falls back again. When the water has risen to its highest level, covering much of the shore, it is at high tide. When the water falls to its lowest level, it is at low tide.

Tides are caused by a combined effect of the gravitation of the moon and the sun, and also by the rotation of the earth. Although the sun and moon both exert gravitational force on the Earth, the moon's pull is stronger because the moon is much closer to the Earth than the sun is. The moon's ability to raise tides on the Earth is an example of a tidal force. The part of the earth facing the moon and the part opposite to it experience a bulge in the ocean called a high tide. Between these parts, the water level falls. This is called a low tide. High and low tides occur alternatively twice every day (on a 12h 25m interval to be more precise) because that is the time moon takes to face opposite sides of the earth every day. The extent of change in water level also changes on a fortnightly basis. During a full or new moon, both sun's and moon's tidal forces act together to cause higher high tides and lower low tides. These are called spring tides. On the other hand, during the first and third quarters, their forces counteract each other to produce low variations in tides. These are called neap tides.

Now we will be looking at how the tidal forces of the sun and moon affect Earth.



(a) Force fields



(b) Resultant force fields

Figure 5.1

Look at the above image showing the force fields due to the sun and moon and the resultant of that force vectors.

- **Black vector** - gravitational acceleration of Moon.
- **Green vectors** - gravitational acceleration at the center.
- **Red Vectors** - resultant vectors.

This clearly shows that

- High tides at two opposite sides of the earth face the moon.
- Low tides at the perpendicular sides.

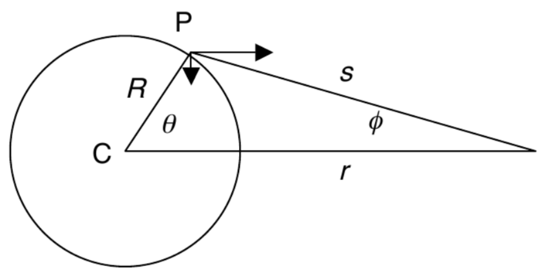


Figure 5.2: Force vectors at point P on earth

The above figure is just showing force vectors(resolved along x and y-components) at a point P on Earth.
We have these 4 equations for forces at point C and point P:

$$F_{P_x} = \frac{GMm}{s^2}\cos\phi \tag{5.1}$$

$$F_{P_y} = -\frac{GMm}{s^2}\sin\phi \tag{5.2}$$

$$F_{C_x} = \frac{GMm}{r^2} \tag{5.3}$$

$$F_{C_y} = 0 \tag{5.4}$$

What we are interested in is the difference between forces at P and C. This is because this $\mathbf{\Delta F}$ is roughly equal to the net force experienced at those points.

$$\mathbf{\Delta F} = \mathbf{F}_P - \mathbf{F}_C \tag{5.5}$$

So,

$$\Delta F_x = GMm\Big[\frac{\cos\phi}{s^2} - \frac{1}{r^2}\Big] \tag{5.6}$$

$$\Delta F_y = -GMm\Big[\frac{\sin\phi}{s^2}\Big] \tag{5.7}$$

Note that $\phi \ll \theta$ as $r \gg R$.
From the cosine rule,

$$s^2 = r^2 + R^2 - 2Rr\cos\theta$$

As $r \gg R$,

$$s^2 \approx r^2\left[1 - 2\frac{R}{r}\cos\theta\right]$$

Take inverse on both sides and doing binomial expansion, we get,

$$\frac{1}{s^2} \approx \frac{1}{r^2} + \frac{2R\cos\theta}{r^3}$$

And also ϕ is very small: just look at the distance between the earth and the moon and the radius of the earth, and you will get to know why I'm saying that ϕ is small and we can use small angle approximation in this by putting $\cos\phi \approx 1$
Now Eq.(5.6) becomes,

$$\Delta F_x = GMm\left[\frac{1}{s^2} - \frac{1}{r^2}\right]$$

and finally, it can be written as,

$$\Delta F_x = GMm\left[\frac{2R\cos\theta}{r^3}\right]$$

Look at figure 5.2, we can write $R\sin\theta = s\sin\phi$
Manipulate this by writing,

$$\frac{\sin\phi}{s^2} = \frac{R\sin\theta}{s^3}$$

As small angle approximation suggests, we can write $s \approx r$
so Eq.(5.7) can be written as,

$$\Delta F_y = -GMm\left[\frac{R\sin\theta}{r^3}\right]$$

The above equations tell us that the X-component of force elongates the planet at equators, and the Y-component of force compresses the planet at the poles. See the below figure,

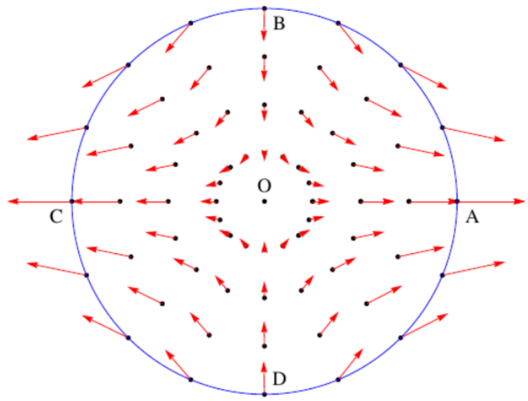


Figure 5.3: Force vector directions

Now clearly from the above equations, we see that tidal forces (difference in force due to which tides arise) is $\propto r^{-3}$, which explains why the effect due to the moon is highest when talking about tides.

Not only tides but also according to a theory, these tidal forces explain the formation of the rings present in Saturn and tell us the minimum distance an object should be from the planet to not disintegrate/the binding forces of the object shouldn't be less than the tidal forces due to the planet. This is termed as Roche Limit.

Trivia

Do you know what happens when two galaxies come close together? If one galaxy is significantly bigger than the other, it will eat it up by first disrupting the smaller galaxy. The process of how this happens is similar to tidal disruption. This is an example of a galaxy merger. The Mice Galaxies (NGC 4676 A&B) are in the process of merging.

Black Holes

Ah finally! Black holes... Who doesn't like black holes? Some of you may have taken this course to know more about black holes. Well rightly so! Black holes are mysterious, we don't know what is inside, etc, etc, perfect to spark human interest! It turns out that with the amount of knowledge that you have gained till now, you can guess pretty cool stuff about black holes (BH for short).

Figure 6.1: One of the first images that comes to our mind when we think of BHs, at least, for those who saw the movie Interstellar (2014). If you haven't seen it yet, stop reading this and go watch it! (This also means that you missed Krittika's screening of the movie :()

6.1 Schwarzschild radius

Strictly speaking, the concept of BHs is a general relativistic thing. But it turns out that you could get some stuff right about them if you use classical Newtonian mechanics. That is what we are going to do!
As you know a BH is called so because even light cannot escape it. So a naive way of estimating the so-called 'point of no return' or 'black hole's radius' or **Schwarzschild radius** $r_{\rm BH}$ would be

to equate the escape velocity to the speed of light.

$$\sqrt{\frac{2GM_{\text{BH}}}{r_{\text{BH}}}} = c$$

$$r_{\text{BH}} = \frac{2GM_{\text{BH}}}{c^2} \tag{6.1}$$

where M_{BH} is the mass of BH.
Even though the above approach is wrong, rigorous general relativistic calculations also lead to the same radius!
The below image is the First image of the Black Hole at the center of Galaxy M87 that has been captured.

Figure 6.2: Black Hole at the center of Galaxy M87.

The image above looks very unimpressive, doesn't it? Well, it's hard to appreciate the above image when there are many excellent graphics/animations of BHs. The image is kinda blurred (stupid photography right?). But capturing the above image wasn't easy. It took a few years, a global team of hundreds of scientists, and petabytes of data to obtain the image. Why is it still blurred then? Well the above BH is a very very tiny object (just 5 billion times more massive than our Sun) in a galaxy far far away (53.49 million light-years). Even though the BH is very massive, its size, the Schwarzschild radius r_{BH} is of the order of 10^{-3} light years. So no surprise that it was very very hard to resolve the object (imagine trying to take a picture of an airplane that is 10 km above you, will you get a clear picture?).

So why did we even bother to take a picture then? One reason: it's pretty cool. Another more scientific reason: it's a direct test of the general theory of relativity (it passed). We can verify the formula of the Schwarzschild radius using the picture! The dark region of the photograph is the event horizon, so the radius of the dark region should be the Schwarzschild radius, right....? No, it turns out to be wrong! Why? Unfortunately, it cannot be dealt with using Newtonian gravity, we need general relativity. Fortunately (or unfortunately for some of you, since you have to read more), we can get some great insights with the help of our current knowledge of celestial mechanics and one equation borrowed from general relativity.

6.2 Shadow of a Black Hole!

So let us phrase the problem properly: given an uncharged, non-rotating BH of mass M, our task is to find the 'shadow of the BH', that is, the **apparent** radius of the dark region as seen by an observer far away from the BH. Strictly speaking, the math that follows this is invalid for most BHs including the imaged BHs from M87 and one from our galaxy since both of them are spinning BHs. A region of space is a shadow if no photons are reaching our eye from that region. By analyzing the trajectories of photons around a BH we can determine the shadow. Mind you, light doesn't travel in straight lines around a BH, gravity is way too high...

Consider a photon currently at a distance of r from the center of BH. We have the **energy conservation**, (the equation comes from general relativistic treatment)

$$\frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} - \frac{l^2 r_{\text{BH}}}{2r^3} = E \tag{6.2}$$

Does it not look very similar to the equation (2.13) ... The only difference from normal Newtonian gravity is the third term, $U(r)$. Here we have $\frac{1}{r^3}$ instead of the usual $\frac{1}{r}$ dependence.

Trivia

The above equation (6.2) is analogous to the energy conservation equation. But notice that E here does not have the dimensions of energy. This is because E here is energy per unit mass of the photon. Photon is a massless particle but for some practical purposes we can consider it to be a particle of mass m_{ph}, where

$$m_{ph} = \frac{E_{ph}}{c^2}$$

$E_{ph} = hf$ is the energy of photon (the above comes from $E = mc^2$).

Look at equation (6.2) again. It is a function of r and $\dot{r}$. Take $v = \dot{r} = \frac{dr}{dt}$. Now doesn't $\frac{1}{2}\dot{r}^2 = \frac{1}{2}v^2$ look like kinetic energy of the system? The rest, $\frac{l^2}{2r^2} - \frac{l^2 r_{\text{BH}}}{2r^3}$ is only a function of r. So we can define a new effective potential $V_{\text{eff}}(r)$ as

$$V_{\text{eff}}(r) = \frac{l^2}{2r^2} - \frac{l^2 r_{\text{BH}}}{2r^3} \tag{6.3}$$

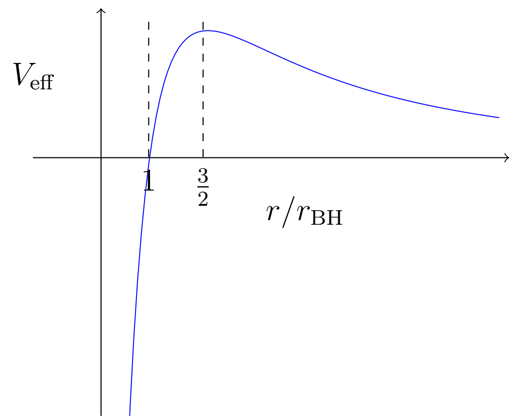


Figure 6.3: Effective potential of photon-BH system as a function of r.

Figure (6.5) shows the plot of V_{eff}. We can see that at $r = \frac{3}{2}r_{\text{BH}}$ we have a point of equilibrium (recall from your JEE days! A maxima/minima is an equilibrium position). However, since $\frac{3}{2}r_{\text{BH}}$ is a point of maxima, the equilibrium is **unstable**. What does that mean? First of all what does equilibrium at $r = \frac{3}{2}r_{\text{BH}}$ mean? If a photon is at $r = \frac{3}{2}r_{\text{BH}}$ and has radial velocity $v = \dot{r} = 0$, the photon has a **circular orbit**! Why a circular orbit, why not stationary? Well duh! It's a photon it must move with velocity c. All its velocity is tangential. Hence the orbit of a photon is circular. But unlike the orbits of planets, this circular orbit is highly unstable. That is, a photon tends to move out or into rather than stay in the orbit. If a photon gains a slight inwards radial velocity, there is no stopping it! It will fall into the BH never to return! If the photon has a slight outward radial velocity, the photon will move away from the BH. The special sphere (radius r_{ph} where photons form circular orbits is called **photon sphere**.

$$r_{ph} = \frac{3}{2}r_{\text{BH}} \tag{6.4}$$

for a non-rotating BH. For a spinning BH there are more than one photon spheres.

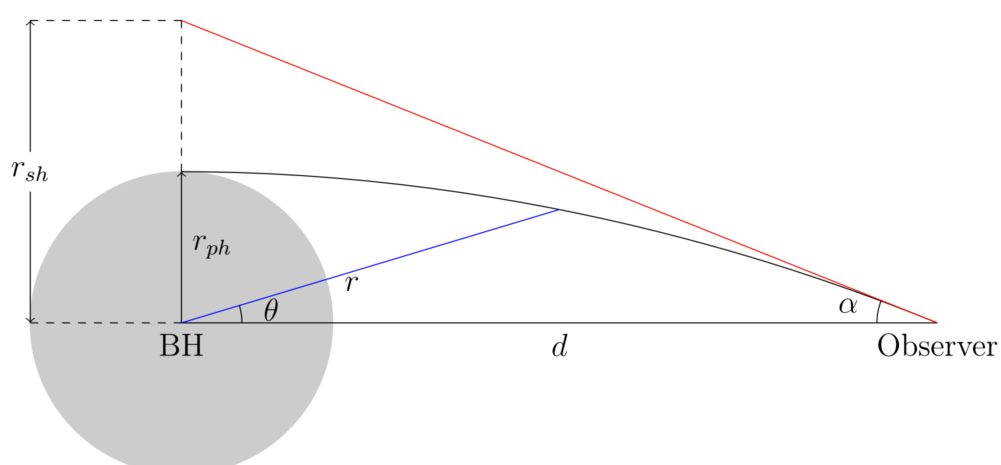


Figure 6.4: The figure shows the trajectory of a photon with 0 initial radial velocity. r_{sh} denotes the size of the dark region perceived by an observer at a distance $d \gg r_{\text{BH}}$.

Note that for the photon-BH system, angular momentum is conserved.

$$r^2\dot{\theta} = l \tag{6.5}$$

θ is defined in the figure above. From equations (6.2) and (6.5), we have

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2E}{l^2}r^4 - r^2\left(1 - \frac{r_{\mathrm{BH}}}{r}\right)$$

Since the initial radial velocity $v = \dot{r} = 0$, we have

$$E = 0 + \frac{l^2}{2r_{ph}^2} - \frac{l^2 r_{\mathrm{BH}}}{2r_{ph}^3} = \frac{2l^2}{27r_{\mathrm{BH}}^2}$$

From figure (6.4) we have

$$r_{sh} = d\tan\alpha$$

$$\tan\alpha = \frac{dy}{dx}$$

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

We have,

$$\frac{dy}{dx} = \frac{\frac{dr}{rd\theta}\tan\theta + 1}{\frac{dr}{rd\theta} - \tan\theta}$$

Here at observation point $\theta = 0$, so

$$\frac{dy}{dx} = \frac{rd\theta}{dr}$$

Also, we had just proved that

$$\frac{dr}{d\theta} = \sqrt{\frac{2E}{l^2}d^4 - d^2\left(1 - \frac{r_{\mathrm{BH}}}{d}\right)}$$

Note that $d \gg r_{\mathrm{BH}}$. So we have

$$\frac{dr}{d\theta} \approx \frac{2d^2}{3\sqrt{3}r_{\mathrm{BH}}}$$

So

$$r_{sh} = d\left(\frac{rd\theta}{dr}\right)_{r=d}$$

Therefore the radius r_{sh} of the dark region or 'shadow region' in a picture of uncharged non-rotating BH is,

$$r_{sh} = \frac{3\sqrt{3}}{2}r_{\mathrm{BH}} \tag{6.6}$$

One last thing that we need to understand properly is: why is the limiting ray shown in figure (6.4) tangential to the photon sphere? That is because it is nearly impossible for regular photons to escape from inside the photon sphere.

Trivia

The orbit of a photon around a BH is a very unique one. It is inherently unstable, unlike the orbits of massive particles (without GR). This is evident from the effective potential of massive particles.

$$V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

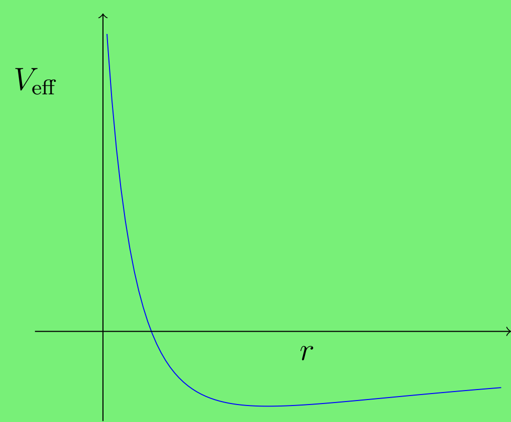


Figure 6.5: Effective potential of a regular two-body system.

Gravitational Lensing

7.1 Introduction

A gravitational lens is a distribution of matter (such as a cluster of galaxies) or a point particle between a distant light source and an observer that is capable of bending the light from the source as the light travels toward the observer. This effect is known as gravitational lensing, and the amount of bending is one of the predictions of Albert Einstein's general theory of relativity. In general relativity, gravity is the result of objects moving through curved spacetime, and everything that passes through, even massless particles such as photons, Thus, treating light as corpuscles traveling at the speed of light, we can calculate deviation in the motion of light. Newtonian physics also predicts the bending of light, but only half of that is predicted by general relativity.

Unlike an optical lens, a point-like gravitational lens produces a maximum deflection of light that passes closest to its center, and a minimum deflection of light that travels furthest from its center. Consequently, a gravitational lens has no single focal point, but a focal line. If the (light) source, the massive lensing object, and the observer lie in a straight line, the original light source will appear as a ring around the massive lensing object (provided the lens has circular symmetry). If there is any misalignment, the observer will see an arc segment instead.

This effect was confirmed in 1979 by observation of the Twin QSO SBS 0957+561.

7.2 Einstein Ring

Consider a spherically symmetric object with a mass M. This object will act like a lens, with an impact parameter r_0 measured from the center of the object. The deflection equation in this case is given by:

$$\phi = \frac{4GM}{r_0 c^2}$$

The distance to the source is $\frac{d_s}{\cos\beta} \approx d_s$, where $\beta << 1$, and d_L is the distance to the lensing mass. It is then a matter of simple trigonometry to show that the angle θ between the lensing mass and the image of the source must satisfy the equation

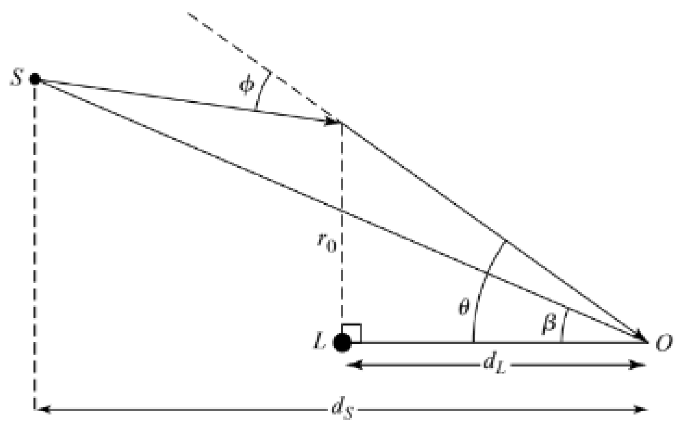


Figure 7.1: Geometry of Gravitational Lensing

$$\theta^2 - \beta\theta - \frac{4GM}{c^2}\left(\frac{d_s - d_L}{d_s d_L}\right) = 0$$

The above quadratic equation indicates that for the geometry shown in the figure, there will be two solutions for θ, and so two images will be formed by the gravitational lens.
If a bright source lies exactly along the line of sight to the lensing mass, then it will be imaged as an Einstein ring encircling the lens. In this case $\beta = 0$.

$$\theta_E = \sqrt{\frac{4GM}{c^2}\left(\frac{d_s - d_L}{d_s d_L}\right)}$$

7.3 Applications of Lensing

The main application lies in the finding of exoplanets. The gravity from stars is far weaker than that of a galaxy cluster, but in some cases, astronomers can still measure lensing from them. "Microlensing" is the effect when one star passes in front of another from our point of view. Stars within the Milky Way don't appear to move quickly from our perspective, but occasionally one crosses another. When that happens, microlensing makes the background star's light appear brighter for a period of weeks to months.

If the closer star is host to exoplanets, those planets alert the microlensing slightly, which astronomers can detect under the right circumstances. Microlensing has let us find smaller planets orbiting farther from their host star than other methods can do easily. Large-scale surveys such as the Korea Microlensing Telescope Network (KMTNet) detect thousands of microlensing events every year, but the number of planets identified this way is still small. Next-generation observatories like the Nancy Grace Roman Space Telescope (NGRST) will potentially detect even more

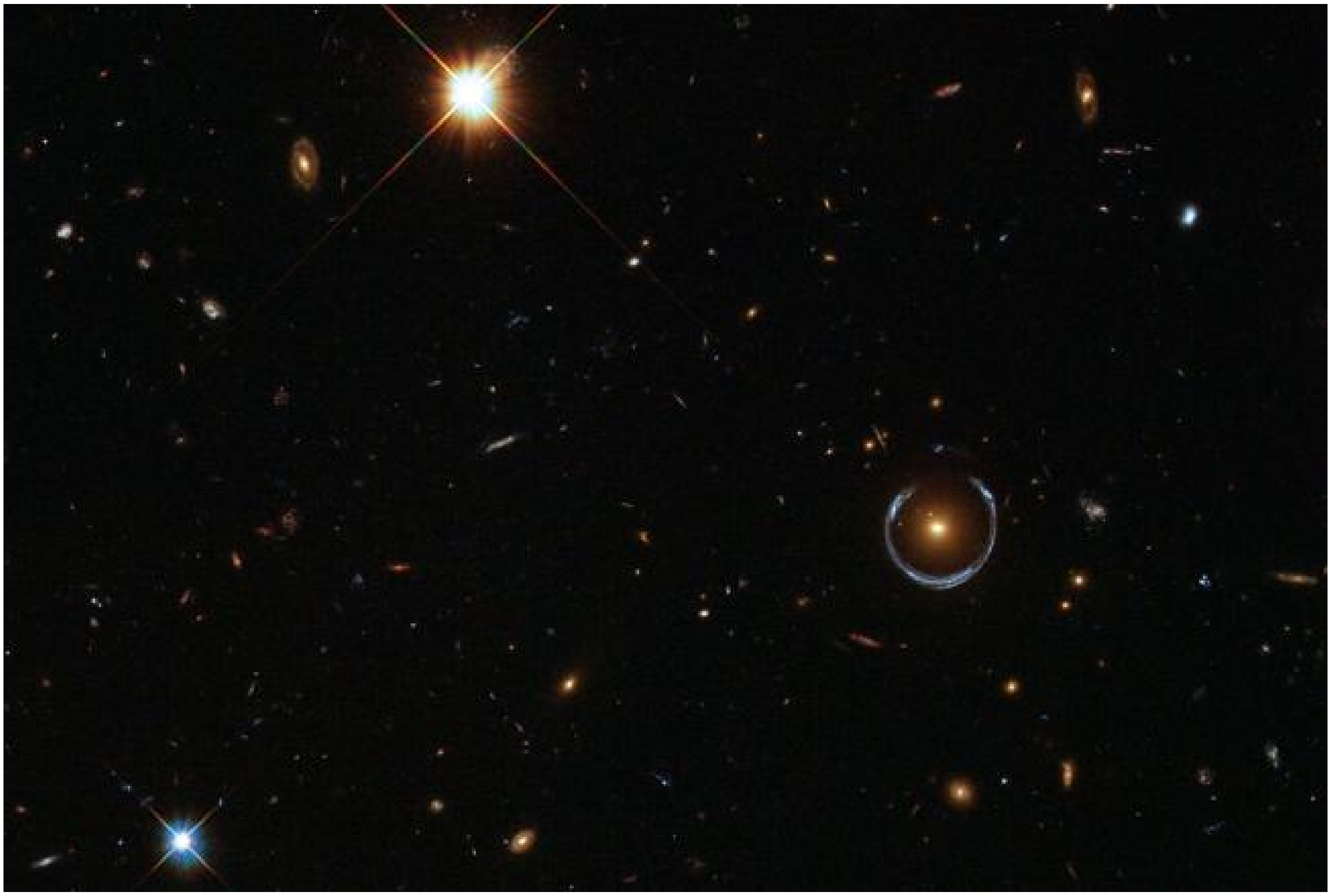


Figure 7.2: The ring in this picture is created by gravitational lensing due to the red galaxy at its center. It distorts the image of a distant blue galaxy. The magnification from the lens lets us see the blue galaxy, which would otherwise be too faint.

in the coming years. **Look at these links below each image to see the animation and get a feel of what's happening**

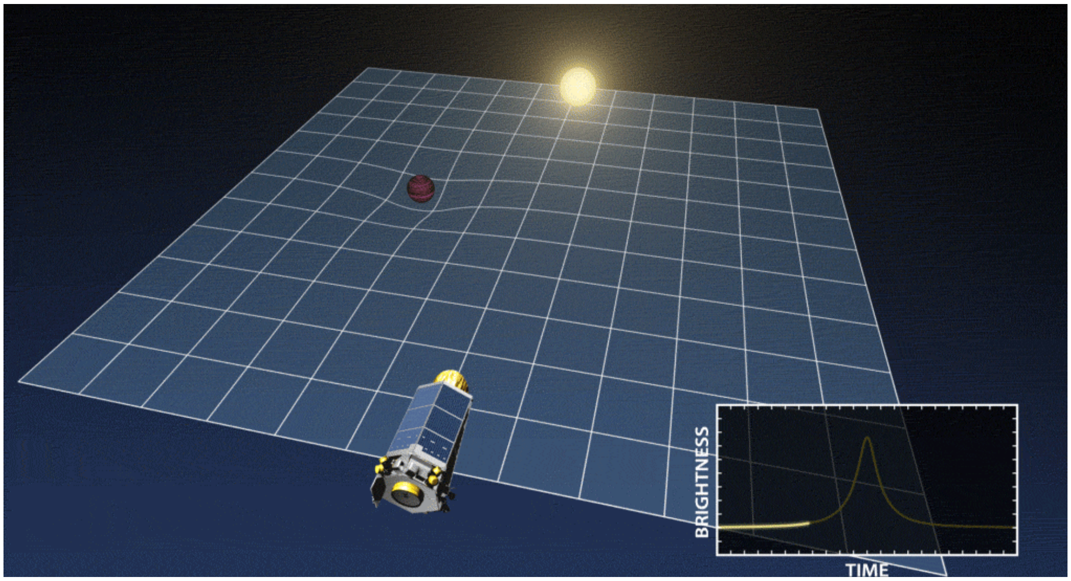


Figure 7.3: Gravitational microlensing of the light of a distant background star by a passing rogue exoplanet
https://en.wikipedia.org/wiki/File:Gravitational_lens.gif

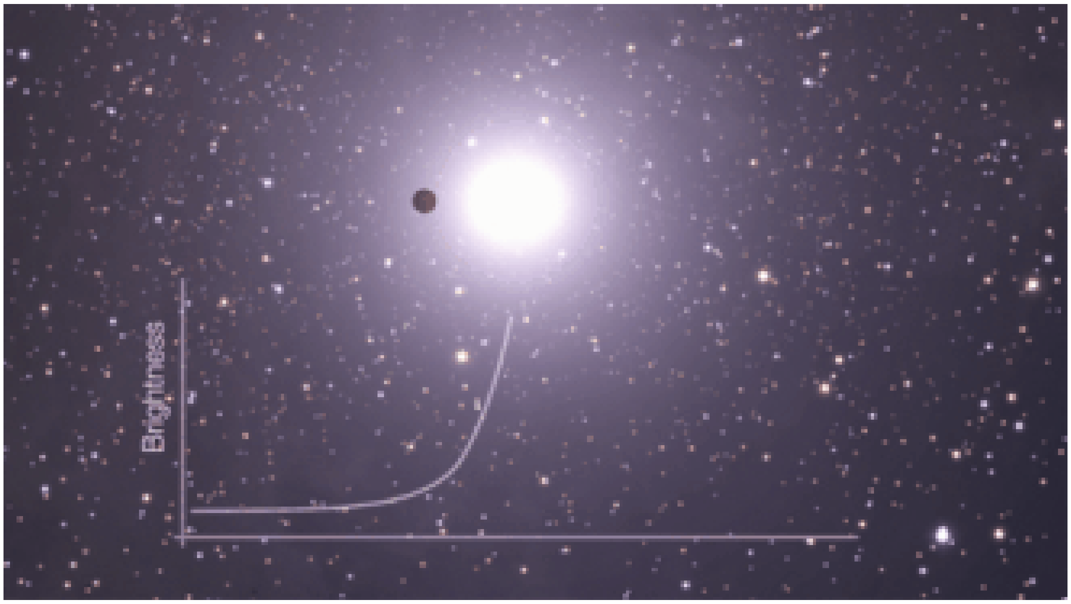


Figure 7.4: Gravitational microlensing of the light of a distant background star by a passing exoplanet with a host star
https://en.wikipedia.org/wiki/File:Microlensingexoplanet.gif

Further Reading

1. An introduction to mechanics by Kleppner ("Central Force motion" for two-body system and "Non-inertial system and Fictious force" for Tides).
2. Celestial mechanics in Fundamental Astronomy by "Hannu Karttunen".
3. "Introduction to Modern Astrophysics by Carroll and Ostlie" for Celestial mechanics and also can be referred for General Relativity at the starting level.
4. You can refer to Sir. Leonard Susskind General Relativity playlist, if someone wants to go deeper in GR.
5. The Circular Restricted Three-Body Problem by Richard Frnka.
6. Poincare and the Three Body Problem by Barrow-Green, June.
7. Astronomy Principles and Practice by A.E Roy and D.Clarke for Hohmann Transfer orbit
Most books mentioned above can be referred to for a basic introduction to other topics of astronomy/astrophysics as well.