Stellar Structure and Evolution

Learners' Space Astronomy

Contents

Standard Notations

Solving a star involves some complex differential equations. Some standard notation is used:

- r - distance from center of the star.
- m $= m(r)$ - mass inside a sphere of radius r centered at the center of the star.
- R - radius of the star
- M - total mass of the star
- $\rho = \rho(r)$ - density at that position
- $l = l(r)$ - the luminosity (energy coming out per unit time) at distance r from center
- L - total luminosity of the star
- P = P(r) - total pressure at r
- P_{gas} - pressure due to gas
- P_{rad} - radiation pressure
- T = T(r) - temperature at r

Introduction

Stellar Astrophysics is a branch of physics + astronomy that deals with stars: their birth, structure, evolution over time and death. Stars are very complex and fascinating objects.

Given the total mass of a star (isolated), its (almost) entire structure and evolution can be predicted.

The most critical force in astronomy is gravitation. Gravitational force is important in stars also because of the unimaginably high masses ($\sim 10^{30}$ Kg). All the stars 'try' to fight back gravitation, but in the end, gravity always wins. How, you may ask? The Core of a star generates tremendous heat, which leads to the accumulation of pressure within the surrounding gas. This pressure counteracts the force of gravity, creating a state called hydrostatic equilibrium. This equilibrium is sustained by the interplay between the gravitational pull that draws the star inward and the outward pressure that pushes against it (note that this is not the same as radiation, which plays a much smaller role in most stars). As long as this delicate balance persists, the star maintains its stability. The more massive the star, the faster it succumbs to gravity, the shorter its lifetime.

Stars are formed in the dense regions of colossal gas clouds. The matter collapses on itself, releasing the loss in gravitational potential energy as kinetic energy. This kinetic energy manifests in the form of temperature and pressure of the gas, slowing down mass collapse. But this kinetic energy leaks due to radiation, and the Core collapses further. After a certain mass is accumulated, the Core becomes hot enough for nuclear fusion (H-H fusion to give He) to start. This nuclear fusion maintains the temperature (and hence pressure) of the star, and a state of quasi-static equilibrium is reached. Every star spends almost its entire life in this stage: burning H in the Core. This stage is called the main sequence period. Only the Core of the star is hot enough for a nuclear reaction to take place. Fast forward some billion years (it could be millions to hundreds of billions, depending on the mass). The Core is now running low on fuel, and the Core starts to collapse. Now the shell just surrounding The Core becomes hot enough for fusion. During this process, the outer layers of the star expand (by a lot!!).

After a specific time, the Core becomes hot enough for a (He-He$\rightarrow$C) fusion reaction. But this does not sustain the Core for long as the amount of He is small. Now the processes become complex. All the stars shed their outer layers, and only a dense core remains. This Core can be of three types (depending on mass):

- White Dwarf
- Neutron Star
- Black Hole

These three are the end states of stars. These are very dense objects. Imagine mass more than our sun inside a 10 Km radius ball! That is a neutron star.
Stars come in different kinds. One might ask why the difference in the properties of stars exists. There are a multitude of parameters on which our observations of stars depend:

- Variation in Stellar Brightness:
  - Variation in distance from us.
  - Variation in intrinsic Brightness.
- Variation in Stellar Colour:
  - Differences in Surface Temperatures (Note: Stars do not have a fixed defined surface, except neutron stars, as they are not made of solids or liquids. We refer to the Photosphere of a star as its surface)

Among various factors that govern the stellar brightness and stellar colour, two are highly dominant:

- Total mass of the star
- Age of the star

As stars age, their global properties, such as radius, luminosity, and surface temperature, change. The internal structure of stars also changes with age, accompanied by a corresponding change in the internal chemical composition. Stars evolve very slowly. Thus, we can consider stellar evolution to be quasi-static. (Except at certain short-lived phases of a star's life, typically towards the beginning or the end stages of the star's life)

There are 4 "global parameters" we use to characterize stars:

1. Mass
2. Radius
3. Luminosity
4. Effective Temperature

Stars emit continuum radiation at all wavelengths. The emission spectrum closely resembles the spectrum of a blackbody very closely. (The blue curve is the best-fit curve which resembles the blackbody emission curve quite closely.) Thus, we could use blackbody conditions to approximate most of the stars' parameters.

Definition: The temperature of a blackbody whose radiation would mimic the spectrum of a star most closely is called the **Effective Temperature (T_{eff})** of the star.

$$\boxed{L = 4\pi\sigma R^2 T_{eff}^4}$$

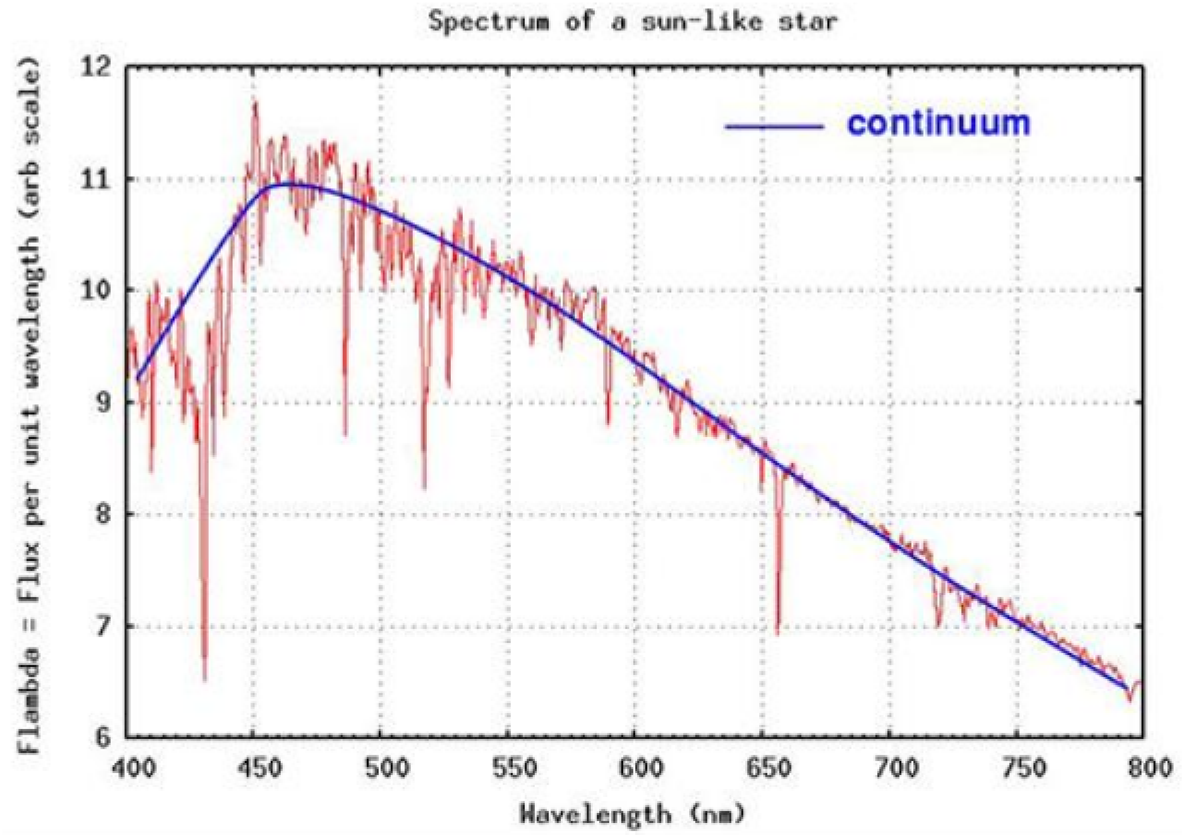


Figure 2.1: A graph showing the brightness flux recorded of a sun-like star

2.1 Basic Assumptions

The following assumptions are made to help approximate star properties:

1. **Isolation:** A single star may be considered as isolated in empty space, so that its structure (and evolution) depends primarily on its internal processes only. Why?

   - Single stars: The given assumption is true for all single stars. For instance, Proxima Centauri: the closest star to the Sun is at 4.3 ly from the Sun. The diameter of the sun is 1.47x 10^{-7} ly. Thus,

   $$\frac{L}{D} = \frac{4.3ly}{1.47\times10^{-7}ly} = 2.925 \times 10^7$$

     Thus, the effects of Proxima Centauri on the sun can be easily neglected.

   - Binary stars: The above condition holds true for most part of a star's life. However, in close binaries (refer definitions page), it may happen that the gravitational and radiation effects of one star may strongly influence the structure and evolution of the other star in the system.

2. **Spherical Symmetry:** We can assume our stars to be spherically symmetric. For instance, the sun bulges by only 10 km at equator, compared to its poles. The primary cause for departure from spherical symmetry is due to stars undergoing very fast rotation or having very strong magnetic fields. This assumption helps reduce the complexity of the mathematics

and reduces the problem to a one-dimensional problem (Only dependent on 'r.' Dependence on 'θ' or 'ϕ' is immediately eliminated)

2.2 Chemical Composition of Stars

Stars are mainly made up of Hydrogen and Helium, along with certain elements in a nominal amount. All other elements apart from Hydrogen and Helium, are termed as "metals" in astrophysics. Thus, we define a quantity, 'Mass Fraction' of a species as:

$$X_i = \frac{\text{mass of species } i \text{ in given mass } m}{m}$$

The following nomenclature is used in general:

$$X \equiv X_H; Y \equiv X_{He}; Z \equiv X_{metals}$$

Thus, we have $X + Y + Z = 1$. **Example:** for the sun, we have X≈ 0.70, Y≈ 0.28, Z≈0.02.

The Basic Equations

The stars spend most of their lives in the so-called 'main sequence,' where the stars burn hydrogen in their cores. Main sequence is a slowly evolving stage, so, it can be assumed that the star is in a 'steady state' with tiny fluctuations/perturbations (having very short time scales as compared to the duration of the main sequence). An isolated, static, and spherically symmetric star can be described by the four basic equations:

- **Conservation of mass:** One might argue that the mass is not conserved as there are nuclear processes in the star, but, that loss of mass (into radiation) is just a small fraction of the mass of star ($\frac{L}{Mc^2} \sim 10^{-21} Kg \cdot s^{-1}$)
- **Conservation of energy**
- **Equation of hydrostatic equilibrium**
- **Equation of energy transport**

3.1 Conservation of Mass

Consider a thin spherical shell of radius r and thickness dr inside a Star of radius R.

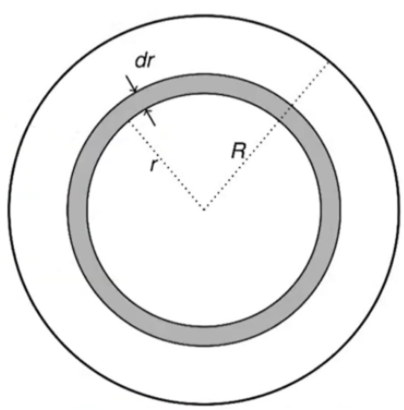


Figure 3.1

The mass contained in the shell is given as:

$$dm = m(r + dr) - m(r) = 4\pi r^2 dr \rho(r)$$

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{3.1}$$

This is called the first equation of stellar structure called **The Mass Conservation Equation**. If the above equation is required, with m as an independent variable, we obtain:

$$\boxed{\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho(m)}}$$

3.2 Hydrostatic Equilibrium

The star does not collapse under gravity or does not blow up due to internal pressures. This is because there is a perfect balance of these forces.
For the thin shell of mass dm,

- inward force of gravity is $\frac{Gmdm}{r^2}$
- Outward force is due to pressure = Δ (Pressure forces at r, r+dr)

In general, P= P_{gas}+P_{rad} (for low mass stars, $P_{gas} \gg P_{rad}$)

For no motion of any layer, i.e. Hydrostatic equilibrium, we have: (for the selected layer)

$$P(r)(4\pi r^2) - P(r + dr)(4\pi r^2) - \frac{GMdm}{r^2} = 0$$
$$-\frac{dP}{(dr)}(dr)(4\pi r^2) - \frac{GM(4\pi r^2 \rho(r) dr)}{r^2} = 0$$

On further simplification, we obtain:

$$\boxed{\frac{dP}{dr} = -\frac{GM}{r^2}\rho(r) = -g\rho(r)} \tag{3.2}$$

3.3 Central Pressure

A crucial question that comes up now is to be able to find the pressure at the centre of the star. (i.e., P_C).

- Pressure vanishes at the surface of the star: P(R)=0 (This is a good exercise)
- Thus, the second stellar equation is now an Ordinary Differential Equation in r, with initial conditions P(R)=0 at r=R.

  $$P\left(R\right) - P\left(0\right) = -P_C = -\int_0^R \frac{Gm\rho(r)}{r^2} dr$$

  An issue arises. We do not know $\rho(r)$, the density distribution inside the star.

- Alternatively, we could solve it in terms of m as the independent variable:

$$\frac{dP}{dm} = \frac{\left(\frac{dP}{dr}\right)}{\left(\frac{dm}{dr}\right)} = -\frac{\left(\frac{Gm\rho(r)}{r^2}\right)}{4\pi r^2\rho(r)} = -\frac{Gm}{4\pi r^4}$$

And integrate it:

$$P(M) - P(0) = -P_C = \int_0^M dP = -\int_0^M \frac{Gm}{4\pi r^4} dm$$
$$P_C = \int_0^M \frac{Gm}{4\pi r^4} dm$$

We do not know how r varies with the mass inside either. But now, we can apply the inequality: r<R

$$P_C = \int_0^M \frac{Gm}{4\pi\ r^4} dm > \int_0^M \frac{Gm}{4\pi R^4} dm$$

$$\boxed{P_C > \frac{GM^2}{8\pi R^4}} \tag{3.3}$$

Thus, we have obtained a lower bound for the central pressure in a star. If we plug in the values of solar mass and radius, to obtain $P_{C,\odot} > 4.4$ x 10^{13} N$/m^2$. The actual pressure at the centre of the sun is about 10^{15} N$/m^2$

3.4 Central Temperature

We need to develop an equation of state, which shall connect the pressure which temperature. Reasonable approximation for the equation of state is the ideal gas law ($PV = NkT$).

Why is $PV = NkT$ valid?

The temperatures inside a star are so high that beneath the outermost layers, all atoms are in ionised state, and we have a plasma of positive ions and free electrons. The thermal energy of the ions and electrons dominate by a large factor any (electromagnetic) interaction potential, which means that we have essentially a population of non-interacting particles- an ideal gas.

But $PV = NkT$ is inconvenient in a stellar context, because it contains the extensive variables V and N. Instead, we convert the equation to one involving only intensive variables: $(V, N) \to (\rho, \mu)$.

We define a new quantity: "Mean Molecular Weight" (μ). This is the average mass per particle in the gas, expressed in terms of proton mass.

The mean molecular weight μ of a gas of N particles and total mass m is thus defined to be:

$$\mu m_p = \frac{m}{N}$$

i.e.

$$\mu = \frac{m}{N m_p} \tag{3.4}$$

Example

For, a fully ionised hydrogen gas of N_{H^+} ions of H^+ and equal number of N_e of electrons,

$$\begin{aligned}\mu &= \frac{1}{m_p}\frac{N_{H^+}m_p + N_e m_e}{N_{H^+} + N_e} \\ &= \frac{1}{m_p}\frac{N_{H^+}m_p}{2N_{H^+}} \qquad (since\ m_e \ll\ m_p) \\ &= \frac{1}{2}\end{aligned}$$

Then, the density of a gas can be expressed as:

$$\rho = \frac{m}{V} = \frac{N(\mu m_p)}{V}$$

The ideal gas law is now:

$$P = \frac{NkT}{V} = \frac{\rho kT}{\mu m_p}$$

Thus, central temperature, T_C is:

$$\boxed{T_C = \frac{P_C \mu m_p}{k\rho_C}} \tag{3.5}$$

3.5 Virial Theorem

The virial theorem in astrophysics (don't get confused with the virial theorem in the celestial mechanics module, that is different) was developed to understand the equilibrium and stability of self-gravitating systems, such as galaxies and star clusters. It provides a mathematical relationship between the average kinetic and potential energies, shedding light on the balance between gravitational forces and thermal energies within these systems. This theorem has been instrumental in studying the dynamics, structure, and evolution of celestial objects

The equation of Hydrostatic Balance with m as the independent variable is:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

Multiplying both sides by $4\pi r^3$ and integrating over the whole star (s: surface and c: centre)

$$\int_c^s 4\pi r^3 dP = -\int_c^s \frac{Gm}{r} dm$$

(applying 'by parts')

$$4\pi r^3 P|_c^s - 4\pi \int_c^s 3r^2 dP = \Omega \tag{3.6}$$

where,

$$\Omega = -\int_c^s \frac{Gm}{r} dm$$

At the surface (r=R), P= 0, and at the centre (r=0), Thus, previous equation (3.6) becomes:

$$\Omega = -3\int_c^s P.4\pi r^2 dr = -3\int_c^s P.dV \tag{3.7}$$

This is the **Virial Theorem**. Now, internal energy per unit volume, 'u' is given by:

$$u = nf\left(\frac{kT}{2}\right) \qquad n = \frac{N}{V} = number\ density, \quad f = d.o.f$$

The degrees of freedom f can be linked to the adiabatic exponent γ:

$$\frac{C_P}{C_V} = \gamma = \frac{f+2}{f} \Rightarrow \ f = \frac{2}{\gamma-1}$$

$$\begin{aligned}\implies u = nf\frac{kT}{2} &= \left(\frac{\rho}{\mu m_p}\right)\left(\frac{2}{\gamma-1}\right)\left(\frac{kT}{2}\right) \\ &= \left(\frac{1}{\gamma-1}\right)\frac{\rho k\ T}{\mu m_p} \\ &= \frac{P}{\gamma-1}\end{aligned}$$

Thus, for an ideal gas, we can write:

$$\begin{aligned}\Omega &= -3\int_c^s P.dV \\ &= -3\left(\gamma-1\right)\int_c^s u.dV \\ &= -3\left(\gamma-1\right)U\end{aligned}$$

Where, U is the total internal energy of the star. Therefore,

$$\boxed{3\left(\gamma-1\right)U + \Omega = 0} \tag{3.8}$$

This is the **Virial Theorem for a star in the hydrostatic equilibrium (under the assumption of ideal gas equation of state)**.

For a fully ionised monoatomic gas, $\gamma = \frac{5}{3}$. Using this,

$$2U + \Omega = 0 \Rightarrow U = -\frac{\Omega}{2} \tag{3.9}$$

Notice that by definition, Ω<0 and U>0, which is consistent with the above equation. Without the ideal gas approximation, the Virial Theorem would look like $\alpha U + \Omega = 0$, where $\alpha \sim 1$. Importantly, the thermal energy of the star has been connected to the gravitational energy of the star.

Negative Heat Capacity

The total energy E of the star is the sum of the thermal energy(internal energy), gravitational energy and possibly the energy generated by nuclear fusion.
Thus,

$$E \approx U + \Omega = -\frac{\Omega}{2} + \Omega = \frac{\Omega}{2} = -U$$

E<0, Which indicates that the star is gravitationally bounded.
Now,

$$\begin{aligned} E \uparrow &\implies \Omega \uparrow (expansion; work\ is\ done\ against\ gravity) \\ &\implies U \downarrow (cooling\ down) \end{aligned}$$

Thus, stars cool down on expansion. Conversely, they contract and heat up, and the total energy decreases.

So, stars have this peculiar property that they have negative heat capacity! They heat up when they lose energy, and vice versa.

Stability of Nuclear Burning

If somehow, the nuclear fuel burning increases in a star, the total energy in the star increases:

$$\begin{aligned} E \uparrow &\Rightarrow \Omega \uparrow (expansion) \\ &\Rightarrow U \downarrow \\ &\Rightarrow T \downarrow (cooling\ down) \end{aligned}$$

Resulting in a decrease in nuclear burning inside the star.

Thus, nuclear burning in a star is self-regulating.

No Universal contraction/ expansion or heating up

In thermal equilibrium, exactly as much energy is lost as the surface as is generated inside the star.

$$\dot{E} = L_{nuc} - L = 0 \ \ (\ L_{nuc}\ is\ the\ rate\ of\ nuclear\ energy\ production)$$

$$\begin{aligned} &\implies \dot{U} + \dot{\Omega} = 0 \\ &\implies \dot{U} = 0, \dot{\Omega} = 0 \\ &since,\ U = -\frac{\Omega}{2} \end{aligned}$$

Thus, a star in thermal and hydrostatic equilibrium can neither globally expand nor globally contract. If one part of a star expands, another must contract in order to conserve the entire Ω.

Similarly, the star cannot globally heat up or cool down. If one region heats up, another must cool down to keep U constant.

3.6 Timescales and Estimations

For any astronomical model/theory it is best to get order of magnitude estimates before refining the models/theories. This to get the scale of things that are being dealt with. All the following calculations are made using solar radius and mass.

3.6.1 Dynamic Timescale

This is an estimate of the time period of small perturbations of a star from its hydrostatic equilibrium. For small perturbations, the force on a mass element at r will be of the order of

$$F = m\frac{d^2\Delta}{dt^2} \sim -\frac{GM^2}{R^3}\Delta r \tag{3.10}$$

$$t_{dynamic} \sim \left(\frac{R^3}{GM}\right)^{\frac{1}{2}} \sim 1h \tag{3.11}$$

3.6.2 Kevin Helmholtz Timescale

It was proposed that the sun radiates energy due to the gravitational collapse. That is, reduction in gravitational potential energy converts into radiation. If this hypothesis were true, the energy radiated by our Sun till now, would be

$$E \sim \frac{GM^2}{R^2} \tag{3.12}$$

Since the energy can be approximated by the luminosity multiplied by time, the age of sun can be estimated as

$$t \sim \frac{GM^2}{LR^2} \sim 10^7 yr \tag{3.13}$$

This contradicts the age of the Earth known from fossil records. So, gravitation is not the only source of energy.

3.6.3 Nuclear Timescale

It is the time scale of burning of the 'nuclear fuel'. Typically,

$$t_{nuclear} \sim \frac{\alpha Mc^2}{L} \sim 10^{10}\, yr \tag{3.14}$$

(α = Fraction of mass that converts to energy. For sun, $\alpha = 0.007$)

This timescale represents the life time of a star. From birth to compact object.

3.6.4 Core Temperature and Pressure

Let us use a simple model to estimate the pressure and temperature at the centre of a star : the constant density model, that is, the density of the star is assumed to be uniform.

$$\rho = \frac{3M}{4\pi R^3}$$

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} = -\rho^2\frac{4\pi G\ r}{3}$$

$$P_C = \frac{4\pi\rho^2 G}{3}\int_0^R r dr = \frac{3GM^2}{8\pi R^4} \sim 10^{14} Pa \tag{3.15}$$

This value is about two orders away from the value calculated from advanced models. This is probably due to the fact that the major contributor of the core pressure is the highly dense core itself which is about ∼0.3M and ∼ 0.2R. Let μ be the effective molecular mass. If the core is assumed to be composed almost entirely of ionised hydrogen, $\mu \approx \frac{m_p}{2}$

$$P_\mu = \rho k_B T \Longrightarrow T_C = \frac{P_\mu}{\rho k_B} = \frac{GM\mu}{2k_B R} \sim 10^7 K \tag{3.16}$$

This value is reasonably close to the solar core temperature calculated from sophisticated models: 1.5×10^7 K. The estimate of temperature does not get effected, due to high density of the core, as much as pressure because $T \sim \frac{P}{\rho}$

Transport of Energy

4.1 Calculation of Molecular Weights

A star contains many elements but H and He are the only abundant ones. In astrophysics, rest of the elements are called metals! The atoms present in the star are in varying stages of ionisation (based on the distance from centre). Since the electron/s are not bound to the ions, they behave as individual particles, which changes the average molecular weight.

Let X_i denote the mass fraction of i^{th} species (different isotopes of an element are considered different), Z_i its atomic number, A_i its mass, and n_i its number density. The number density of the species will be

$$n_i = \frac{X_i \rho}{A_i}$$

$$n_I = \sum_i \frac{X_i \rho}{A_i} = \frac{\rho}{\mu_I}$$

where, n_i and μ_i are the number density and average atomic weights of all the ions respectively. To know the number density of the electrons, we need to know the fraction of electrons that are ionised for each of the species. Let y_i be the fraction of electrons ionised in the i^{th} species. The number density of electrons n_e will be

$$n_e = \sum_i \frac{y_i Z_i X_i}{A_i} \rho = \frac{\rho}{\mu_e}$$

$$\frac{\rho}{\mu} = n = n_e + n_i$$

$$\mu = \left[\sum_i \frac{(y_i Z_i + 1) X_i}{A_i} \right]^{-1} \tag{4.1}$$

where, n is the number density and μ is the average molecular weight. μ_e is a quantity defined for symmetry.

Since the 'metals'/'heavies' (species other than ^{1}H and ^{4}He) are low in abundance they are generally clubbed together. X, Y and Z denote the mass fractions of ^{1}H, ^{4}He and the rest of elements, respectively.

In the cores of most main sequence stars, Z $\ll$ 1, so,

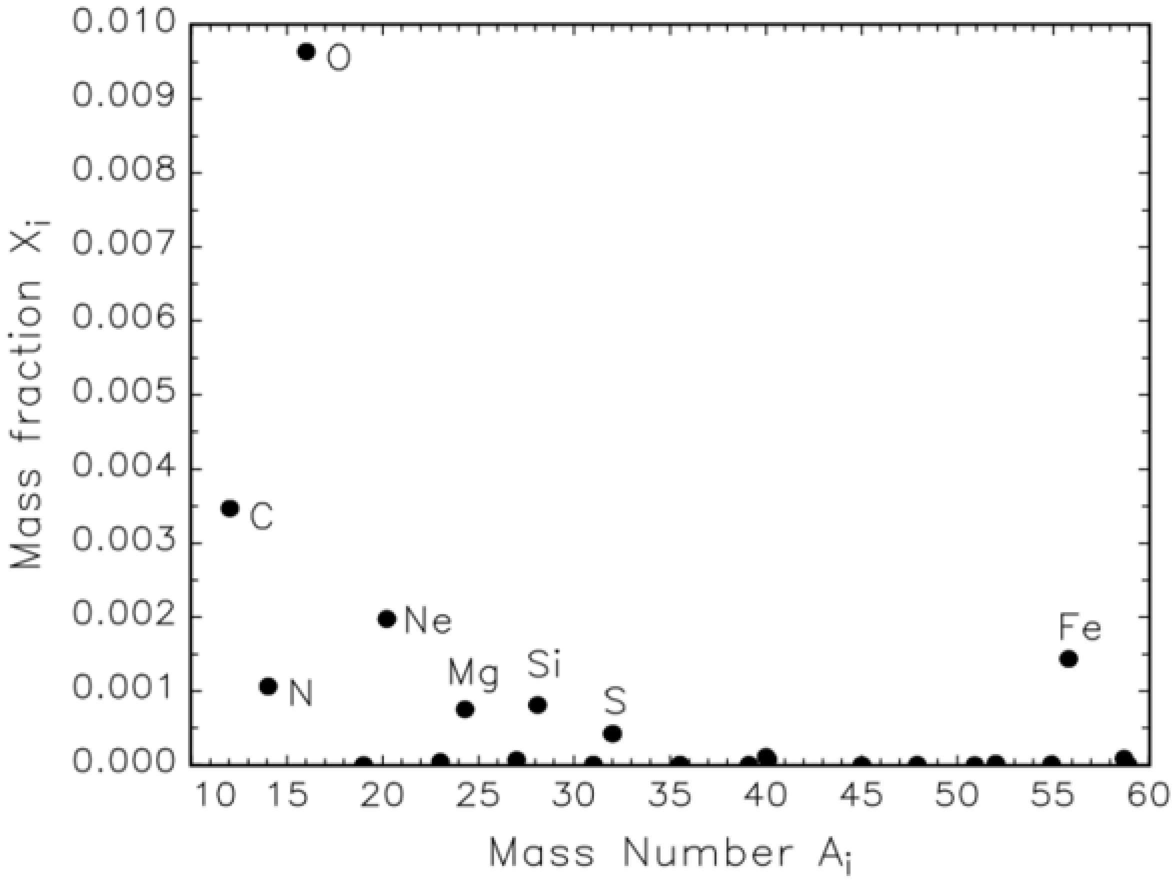


Figure 4.1: Metal abundances for the solar atmosphere vs average elemental mass number

$$\frac{n_e}{\rho} = \frac{1 \times X}{1} + \frac{2 \times (1 - X)}{4} = \frac{1 + X}{2}$$

$$\frac{n_I}{\rho} = \frac{1 + 3X}{4}$$

$$\mu \approx \frac{4}{3 + 5X}\, amu \tag{4.2}$$

4.2 Conservation of Energy

For a star in 'steady state' (main sequence), all the energy generated must be radiated out. Otherwise, it will result in change in temperatures : non-steady state (contradiction to out assumption). Let ε be the amount of energy generated in a star by a unit mass per unit time (due to the nuclear processes etc).

$$l\,(r + dr) - l\,(r) = 4\pi r^2 \rho \varepsilon$$

$$\frac{\partial l}{\partial r} = 4\pi r^2 \rho\ \varepsilon$$

ε depends on the density ρ temperature T (very heavy dependence) and elemental composition $X_i s$.

4.3 Energy Transport

There are 3 modes of energy transfer:

- **Conduction**
- **Convection**
- **Radiation**

Conduction is insignificant in all the stars except for the compact stars (White Dwarfs). Radiation is the dominant mode of transfer in the low mass stars. Convection is a dominant mode of transfer when the temperature gradient is very high. Convection is dominant in the cores of high mass stars and the outer (cooler, highly opaque) layers of lower mass stars.

4.4 Radiative Transfer

This is the dominant mode of heat transfer in the low mass stars. The photons produced in the core do not have a free path outside. They are absorbed, re-emitted and scattered multiple (an understatement) times before emerging out of the star (or a layer of star). Let f(r) denote the radiative flux (energy per unit area per unit time) at distance r from centre.

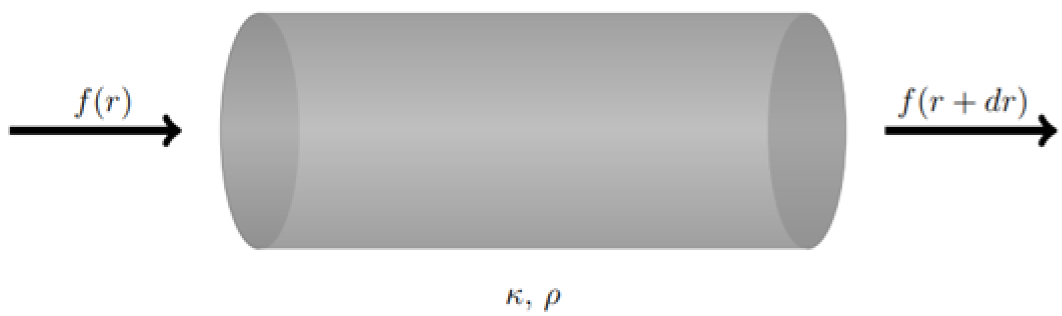


Figure 4.2

$$f\left(r\right)=\frac{l_{rad}}{4\pi r^2}$$

$$f\left(r+dr\right)-f\left(r\right)=-\kappa\rho f\left(r\right)dr\implies df\left(r\right)=-\kappa\rho f\left(r\right)dr$$

$$P_{rad}\left(r+dr\right)-P_{rad}\left(r\right)=dP_{rad}=\frac{df}{c}=-\frac{\kappa\rho f\left(r\right)}{c}dr \tag{4.3}$$

where l_{rad} is the luminosity at r from centre, κ is the opacity (fraction of energy scattered per unit distance and density).

4.4.1 Radiation Pressure

Photons have momentum; hence they can exert force and pressure when they interact with matter. Consider a cavity at thermal equilibrium at a temperature T . The photons inside the cavity exert a pressure of

$$P_{rad} = \frac{u}{3}$$

refer: this

Where u is energy density of radiation,

$$u = aT^4$$
$$a = \frac{4\sigma}{c}$$

Now,

$$\frac{dP_{rad}}{dr} = \frac{4aT^3}{3}\frac{dT}{dr}$$
$$\frac{4aT^3}{3}\frac{dT}{dr} = -\frac{\kappa\rho f}{c} = -\frac{\kappa\rho l_{rad}}{4\pi r^2 c}$$

Thus,

$$\boxed{\frac{dT}{dr} = -\frac{3\kappa\rho l_{rad}}{16\pi a c r^2 T^3}} \tag{4.4}$$

The above equation is **radiation dominated energy transport equation**.

Causes for Opacity

Opacity to radiation is caused by 4 processes:

- **Bound - bound transition** - bound electron absorbs photon and moves to higher state and again de-excites to releasing photon in a random direction.
- **Bound - free** transition
- **Free - free** transition - free electron excites to a different energy free state and de-excites releasing photon in a random direction.
- **Electron scattering** - Compton scattering and Thomson scattering

An Estimate...

Let us try and estimate the time taken by a photon to reach the surface of Sun from its core. Let us assume that there is only electron scattering (for simplicity). Let σ_e be the Thomson scattering cross-section of an electron and n_e (assumed to be constant) be the electron number density. Let λ be the mean free path of a photon.

For a scattering to happen,

$$n_e\lambda\sigma_e = 1 \implies \lambda = \frac{1}{n_e\sigma_e}$$

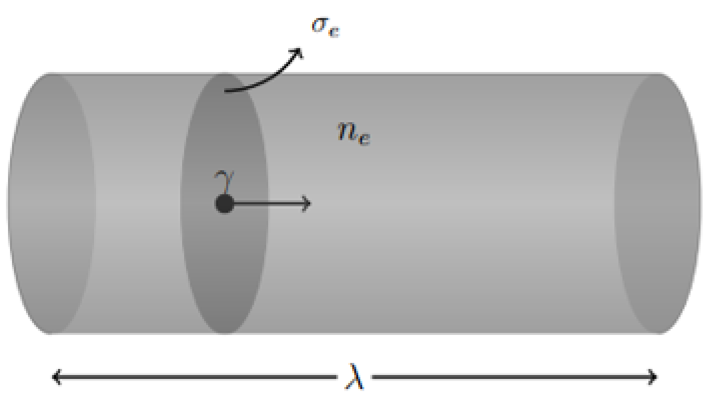


Figure 4.3

Let d_N be the rms displacement after N collisions.

$$\langle d_N^2 \rangle = \langle d_{N-1}^2 \rangle + \lambda^2 + \langle 2\lambda d_{N-1} cos\phi \rangle = \langle d_{N-1}^2 \rangle + \lambda^2$$

$$d_N = \sqrt{N}\ \lambda \tag{4.5}$$

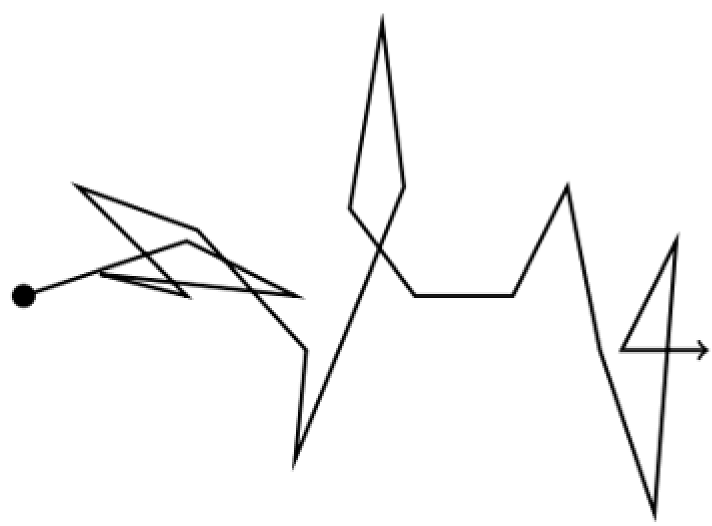


Figure 4.4: An image depicting a random motion of a photon within the sun

Time taken to traverse a displacement of d_N, t_N:

$$t_N = \frac{N\lambda}{v} = \frac{d_N^2}{\lambda v} = \frac{n_e \sigma_e d_N^2}{v}$$

$$t = \frac{3M\sigma_e}{4\pi R\ m_p c} \sim 1000yr \tag{4.6}$$

4.5 Convective Transfer

Convection is a very complex process. So, unlike radiation and conduction, there is no universally accepted theory of convection. But this equation is accepted for convection : (strictly speaking it should be an inequality but it's almost equal.) (The derivation is directly from the Adiabatic Gases)

$$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dP}{dr} \tag{4.7}$$

4.6 Auxiliary Equations

Now we have four equations of stellar structure(Namely: Conservation of Mass, Equations of hydrostatic equilibrium, Conservation of Energy, Equations of Energy Transport). But those equations have 7 variables, so, we need 3 more. The three are :

1. Equation of state - An equation relating the pressure P , temperature T , number density n, etc. Like $PV = Nk_BT$ (equation of state for ideal gas).

2. Opacity - We need opacity κ as a function of density ρ, temperature T and local composition X_i

3. Nuclear Energy generation rate (ε) - nuclear energy generation rate is very sensitive to temperature. It is a function of T , density ρ and composition X_i.

4.7 Boundary Conditions

One of the obvious boundary conditions is

m(0)=0

But the others are not so straightforward. Oh yes, we could take that the star 'ends' when the density ρ or the pressure P falls to zero, but, this turns out to be really bad one as the density or pressure doesn't fall to zero (or nearly zero) for several observed radii (observed radii is the radius of part that we see as an opaque ball). So we normally end our model at photosphere/chromosphere, do modelling of stellar atmosphere and match the boundary conditions.

Conclusion

With these equations and boundary conditions one can solve a star (not an easy task, computationally!). The composition X_i, opacity κ and the nuclear energy generation rates ε are nearly constant over small timescales but start showing variations over Kelvin-Helmhotz timescales. So, we must keep updating their values. Also, they also vary spatially.

Values of ε, κ are obtained as tables and are used in computing the stellar models. It takes huge huge amount of computation to come up with a good model!

Photosphere and Chromosphere

The photosphere is the outermost layer of a star that emits light. It is the region where the plasma (ionized gas) in the star becomes opaque to radiation. This opacity is due to the interactions between photons (light particles) and the charged particles in the plasma.

The depth of the photosphere is often defined by the point at which the optical depth reaches approximately 2/3. In other words, if you were to take a photon's path from the core of the star to the surface, about 50% of the photons would escape without being scattered or absorbed by the plasma when they reach the photosphere.

The photosphere is the visible surface of a star, and it is the layer from which the majority of the star's light is radiated into space. It is also the layer where features such as sunspots and granules (convection cells) can be observed on the surface of the Sun.

The chromosphere is the second layer of a star's atmosphere, located between the photosphere and the corona. It is often observed as a reddish layer, particularly noticeable during a total solar eclipse. While commonly associated with the Sun, chromospheres have also been observed in other stars. In some larger stars, the chromosphere can be a significant portion of the entire star and plays a crucial role in the dynamics of the stellar atmosphere. It is an area of interest for studying phenomena like solar flares and prominences. Exploring chromospheres in stars beyond our Sun provides valuable insights into their atmospheric properties and activities.

Stellar Evolution and HR Diagram

5.1 Preliminaries

Astronomy has many terminologies that are defined for convenience (or historically).

Magnitudes

One of the most common measurements in astronomy is the flux. Flux is the power per unit area received from the source. But astronomers use the magnitude scale to talk about fluxes. Consider two astronomical sources with fluxes f1 and f2. The apparent magnitude of a star (m) scales as

$$m_1 - m_2 = -2.5log\left(\frac{f_1}{f_2}\right) \tag{5.1}$$

Apparent magnitude of a star only indicates the relative power of the sources. The absolute scale is given by the absolute magnitude (M). Absolute magnitude is the apparent magnitude from a distance of 10 pc (1 pc = 3.26 ly = 206265 AU)

$$m - M = 5log\ d\ - 5 \tag{5.2}$$

(d is in pc)

Spectrum of Stars

Different physical processes involve different energies. So it is essential to observe a body in the entire electromagnetic spectrum to understand all the physical processes going on. To get more information, even the visible (and surrounding) spectrum is broken down into bands. U, B, V, R, I.. etc are the most common ones (B - blue, V - visible, R - red). The magnitude data of objects is collected in each of these bands.

Apart from this, for stars, we measure its spectrum, that is, E_λ (the energy in a small wavelength band dλ) vs λ (the wavelength of light). It is observed that all the stars fit very well to Planck's blackbody distribution.

The corresponding temperature of the best fit blackbody curve is the effective temperature of the star T_{eff}. We define a quantity called as colour. There are many definitions for colour, but

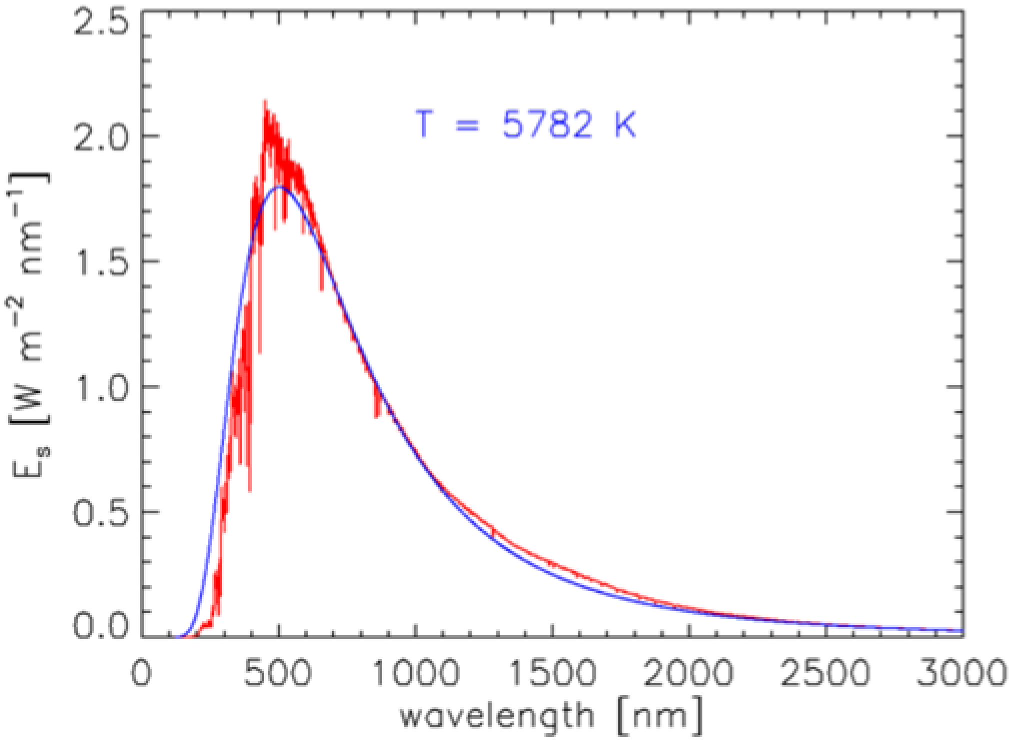


Figure 5.1: The red curves is the observed solar spectrum. The sharp lines are the spectral lines (ones that go down are absorption and those that go up are emission lines). The blue curve shows the best fit blackbody curve

the most common one is B - V. Here B and V denote the apparent magnitude in B and V bands respectively. Colour is a measure of the effective temperature of a star, 'redder' a star, lower its effective temperature.

Doppler Effect

Again, this is a very important phenomenon to extract information. Similar to sound, the wavelength of light changes when there is relative motion between the source and observers.

$$\frac{\lambda_{observed}}{\lambda} = \sqrt{\frac{c+v}{c-v}} \Longrightarrow z = \frac{\Delta\lambda}{\lambda} \approx \frac{v}{c} \quad (for\ v << c) \tag{5.3}$$

where v is the recessional velocity of the source (relative velocity away from us). $\frac{\Delta\lambda}{\lambda}$ is called as the redshift z.

Spectra of stars have many absorption and emission lines. These spectral lines give information about the chemical composition of stars. Studying the shifts and broadening of those lines (due to the Doppler effect) gives a ton of information about both exterior and interior of stars.

Spectral Classification of Stars

Stars are now classified according to multiple methods. A common one out of them is the Morgan-Keenan system. The Morgan-Keenan (MK) spectral classification system is used to categorize stars based on their spectral characteristics. It assigns a spectral type to stars based on their temperature, ranging from O (hottest) to M (coolest), with subclasses from 0 to 9 within each type. It also includes a luminosity class (I-V) to indicate star size and brightness.

5.2 The HR Diagram

Hertzsprung-Russell diagram (HR diagram) is a plot of luminosity of stars vs colour or temerature. Most stars lie grouped together in certain regions of the diagram. The positioning of a star on the diagram is reflective of where it is in the evolution cycle, and was one of the first hints of the first hints astronomers had that stars are not static but ever evolving.

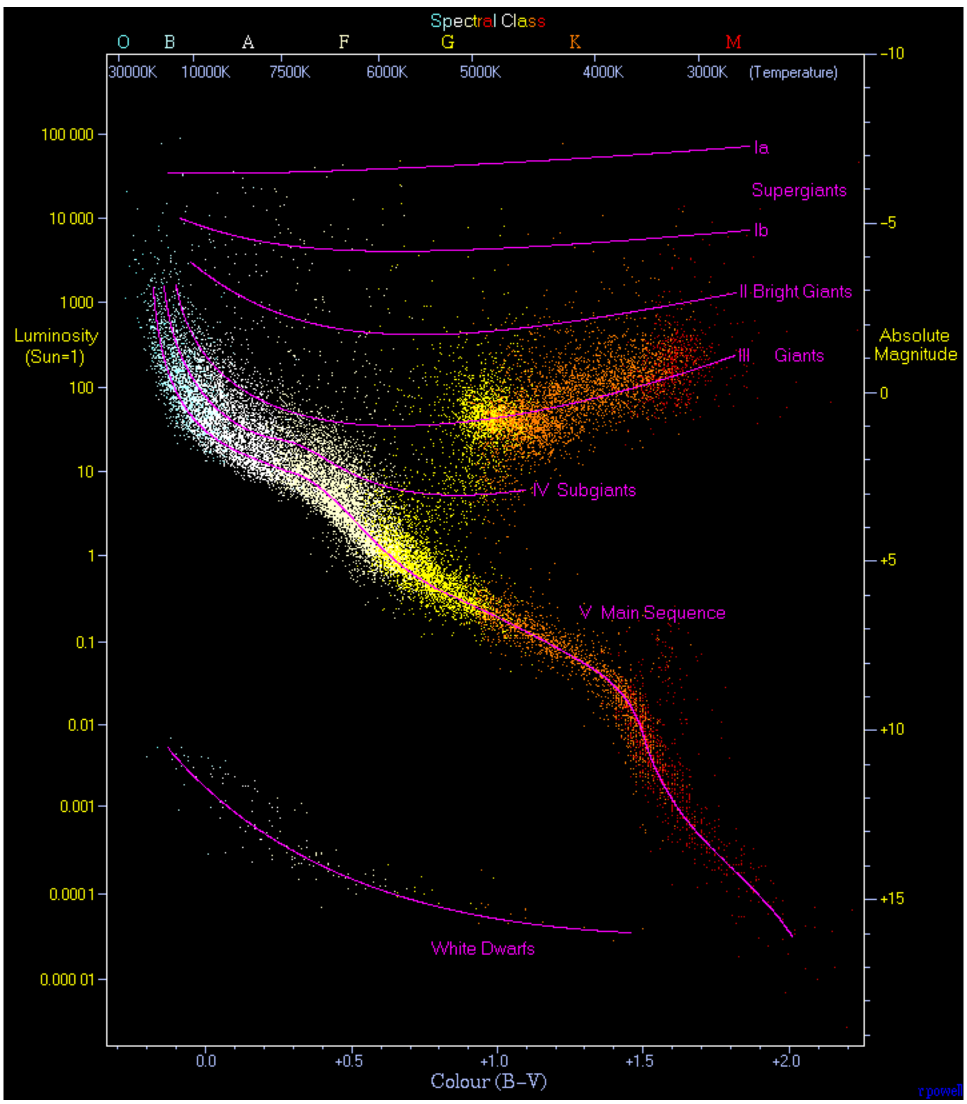


Figure 5.2: The first observation from HR diagram is that, the stars exist only in certain regions of the diagram. Further, all these stars can be classified in groups

Type	Nature
Ia and Ib	Supergiants
II	Bright Giants
III	Giants
IV	Subgiants
V	Main Sequence
VI, sd	Subdwarfs
D	White Dwarfs

Figure 5.3: Each spectral class is subdivided further, for example, O is divided into O0, O1, ..., O9 (number can be decimal till 9.9). O0 is the hottest. Our Sun is a G2-V type star.

5.3 Stellar Evolution

Star formation starts in huge molecular clouds. Matter collapses and the gravitational potential energy that is released is converted into kinetic energy of particles, which, in turn, reflects as temperature. If left to gravity, it would like to collapse the matter till it forms a black hole.

But as the matter collapses the collisions between the particles also increase, increasing the pressure. This gas pressure slows down the collapse. The core part of a 'protostar' is hotter than the exterior layers. After some critical mass is accumulated, the core becomes hot enough to start nuclear reactions. This begins the main sequence phase...

Main Sequence

Main sequence is the phase of the star where it fuses hydrogen to 4He in the core. The nuclear fusion releases energy that maintains the temperature (in turn the gas pressure) of the star, preventing further collapse. Stars spend the longest period of their 'active' life in main sequence. During this period two processes can occur, based on mass. For the low mass stars, like our sun, p - p cycle takes place. For higher mass stars the temperature of the core is hot enough for CNO cycle to take over as the dominant process. Our sun has spent about 4.5 Gyr (Gigayear) in main sequence till now and will spend about 5 more Gyrs. Main sequence is the most stable period of a stars life and properties like mean luminosity and radius hardly change during this period.

The mode of energy transfer in the core of low mass stars is radiation and convection in high mass stars. The envelope follows the opposite trend, lesser the mass, larger the convective zone in the outer layers.

Eventually the star exhausts (nearly) the hydrogen in its core and the main sequence comes to an end....

P-P Cycle

Almost all reactions involve collisions of only two nuclei. So making helium from four protons involves a sequence of steps. In the Sun, this sequence is called the proton-proton chain:

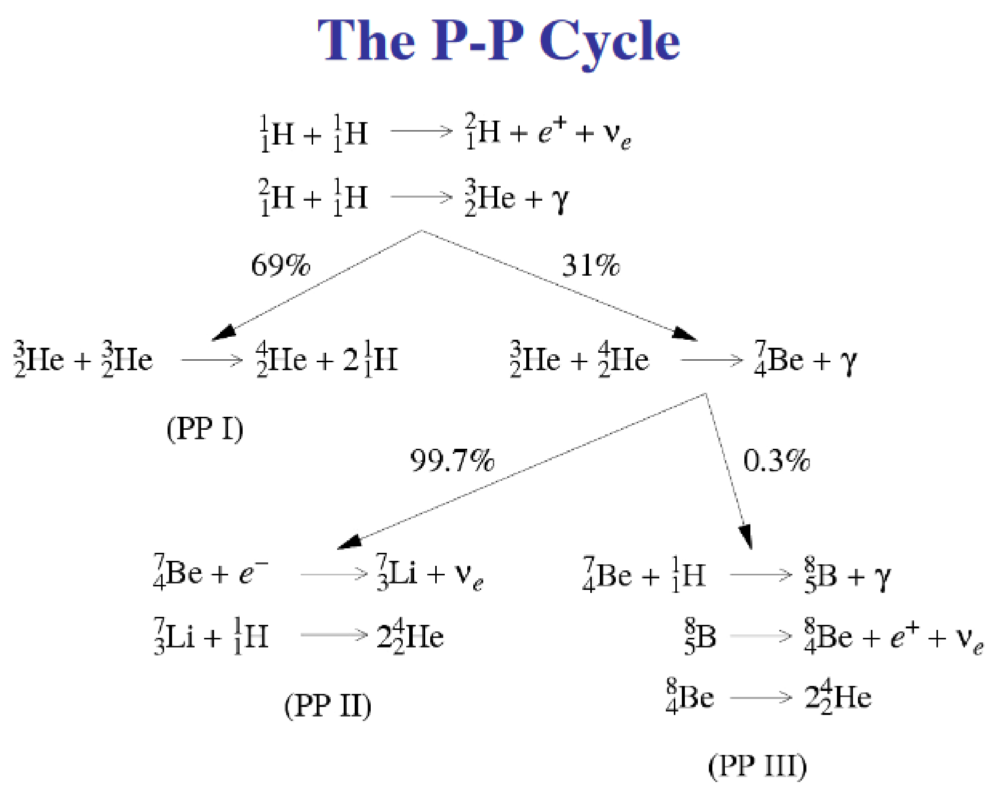
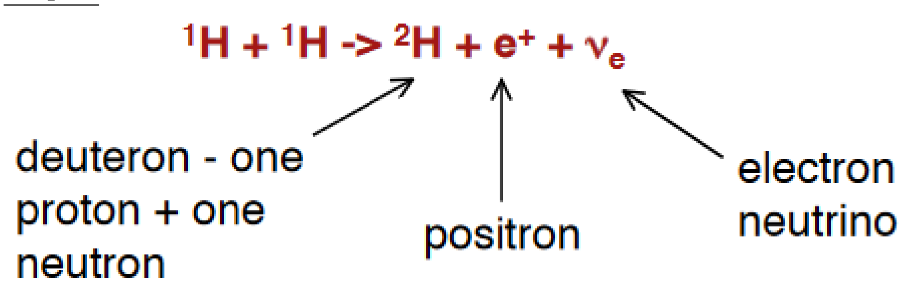


Figure 5.4: P-P cycle

Step 1:



This is the critical reaction in the proton-proton chain. It is slow because forming a deuteron from two protons requires transforming a proton into a neutron - this involves the weak nuclear force so it is slow.

Step 2: $^2H + ^1H \rightarrow ^3He + \gamma$

Step 3: $^3He + ^3He \rightarrow ^4He + ^1H + ^1H$

Results of this chain of reactions:

- Form one 4He nucleus from 4 protons
- Inject energy into the gas via energetic particles: one positron, one photon, two protons
- Produce one electron neutrino, which will escape the star without being absorbed

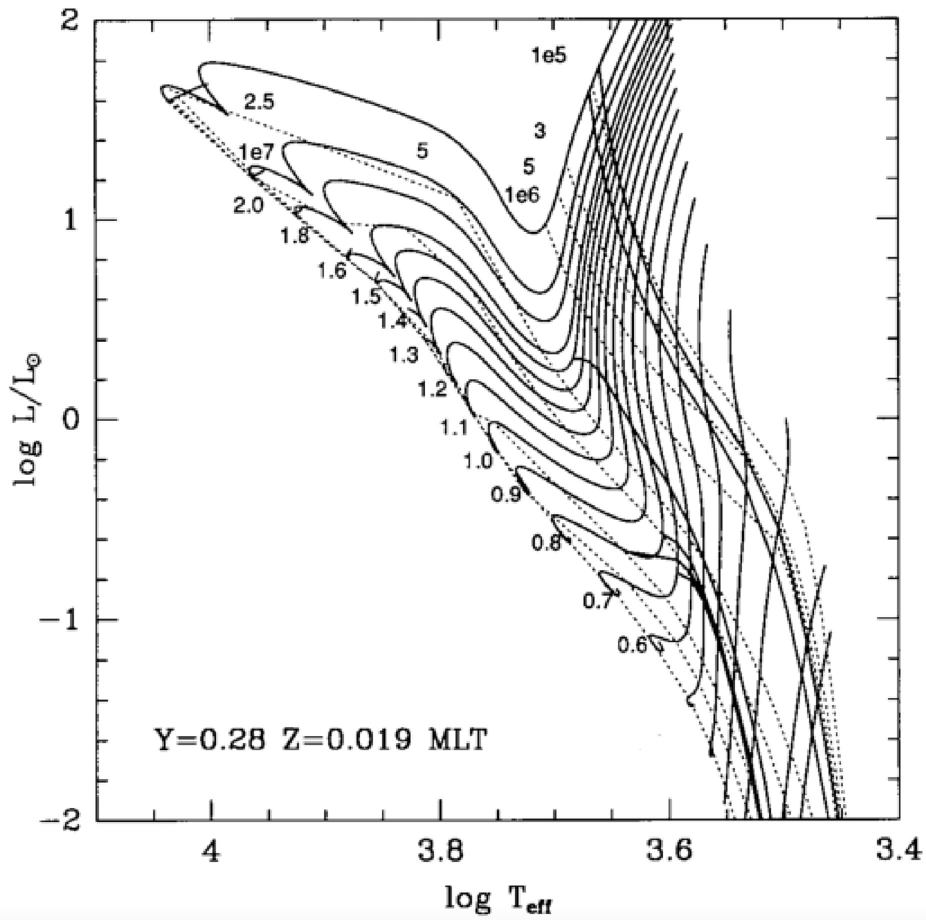


Figure 5.5: Theoretical evolutionary tracks of stars with composition Y = 0.28 and Z = 0.019 for various masses

Sub-Giant

Now the core, made majorly of He and other metals, contracts under gravity and the released gravitational energy expands the outer layers. The radius of the star increases and the outer convective zone deepens, which makes the surface cooler. The layers that are just outside the core heat up enough to start burning H - this is known as hydrogen shell burning ...

Red Giant

After the sub-giant phase, the radius of the star keeps increasing at a nearly constant temperature (small decrease), inreasing the luminosity by a large amount.

Red Clump and Horizontal Branch

After the red giant phase, the core becomes hot enough ($\sim 10^8$ K) to start He burning, triple alpha reaction to produce carbon C. This reaction is extremely sensitive to temperature. The temperature increases and luminosity decreases. The star settles in a '2^{nd} main sequence' called red clump (for high metalicity stars) and the horizontal branch (for the low metalicity stars). But this is very short lived as compared to the main sequence. Most of the luminosity is still produced by the H buring shell.

Asymptotic Giant Branch and Further

The He will be exhausted in the regions that are hot enough. Again, the core will contract and the envelope expands. The shell of He surrounding the C-O core starts burning (even shell H burning goes on)

After this, the process depends on the mass of star. For stars with mass more than $8M_{sun}$, carbon burning starts. Burning of each element has a cutoff mass. For heavy enough star, this continues till ^{56}Fe, the most stable nucleus. The time scale of burning heavier and heavier elements becomes shorter and shorter as the abundances of heavier elements is small (even though they are produced via fusion).

End State

For low mass stars, $< 8M_{sun}$, the envelope keep expanding. At some point it will no longer be bound to the star. So, the star effectively sheds about 50 % of its mass to form a planetary nebula with an extremely dense core left at its center. Further, based on mass, the core can be of two types : White Dwarf and Neutron star. If the birth mass of star $< 3M_{sun}$ the mass of the core left will be $< 1.4M_{sun}$ and the final state will be a white dwarf. If the left over core is $> 1.4M_{sun}$ (birth mass $> 3M_{sun}$), it will form a neutron star.

A white dwarf is a compact object with radius of the order of Earth's. The collapsing gravity is opposed by electron degeneracy pressure. A white dwarf has a high temperature $\sim 10^5$ but low luminosity $\sim 10^{-3}L_{sun}$. An isolated white dwarf keeps cooling for billions of years and becomes very very faint

If the core has mass $> 1.4M_{sun}$, even the electron degeneracy pressure can't stop the compression. The core collapses and forms a neutrons. This core keeps taking in mass but at some point, the neutron degeneracy pressure kicks in and 'reflects' back the matter trying to collapse in. This causes an explosion - a supernova. A supernova releases extremely high amount of energy $\sim 10^{45}$J. The density of a neutron star is of the order of nuclear density and temperature $\sim 10^9$K. The left over core has a size $\sim$ 10 Km. The limit of $1.4M_{sun}$ is called the Chandrasekar limit.

If the birth mass $> 25M_{sun}$, the core is so dense that even neutron degeneracy cannot hold gravity and it collapses to a black hole.

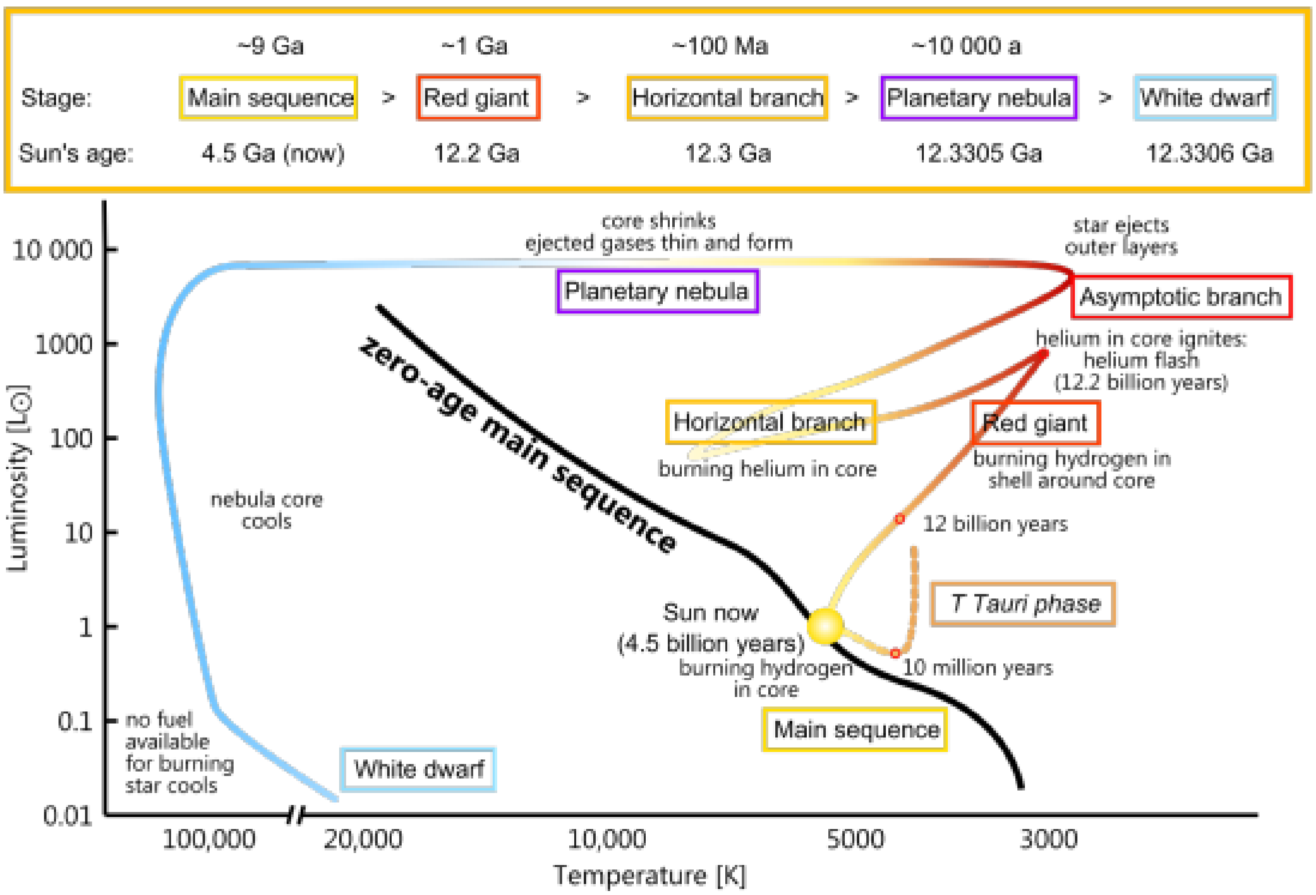


Figure 5.6: Evolution of our sun

Further Readings and References

Hope you enjoyed this module. For further reading,

- More on Energy Transport in Stars: https://www.astro.ru.nl/~onnop/education/stev_utrecht_notes/chapter5-6.pdf
- https://sites.astro.caltech.edu/~george/ay20/Ay20-Lec7x.pdf
- https://en.wikipedia.org/wiki/Stellar_classification
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- Info on triple alpha process: http://large.stanford.edu/courses/2017/ph241/udit2/
- Info on triple alpha process: https://en.wikipedia.org/wiki/Triple-alpha_process
- Info on CNO: https://astronomy.swin.edu.au/cosmos/c/cno+cycle
- Book: An Introduction to Modern Astrophysics (Bradley W Carroll, Dale A Ostlie)