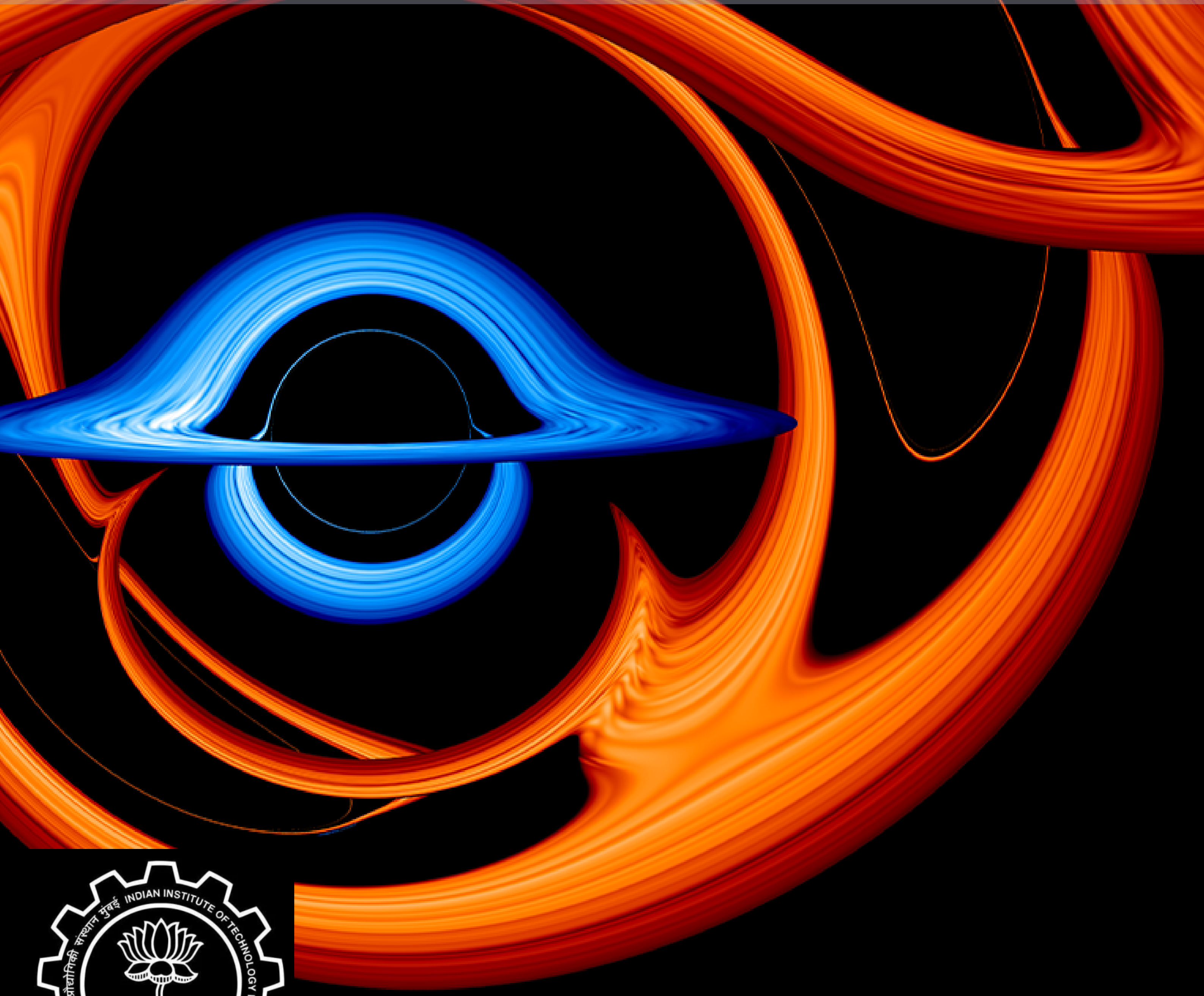
Visualizing Geodesics In Schwarzschild SpaceTime

Krittika Summer Project 2025

Rudra Arya



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1. Special Relativity

The Newtonian view of physics, although immensely successful, fails when objects approach the speed of light. In such regimes, we enter the domain of `Special Relativity`, developed by Albert Einstein in 1905. This chapter outlines the conceptual and mathematical structure of the theory, focusing on inertial frames, Lorentz transformations, and their counterintuitive but experimentally verified consequences.

1.1 Foundations of Special Relativity

Postulate 1: The laws of physics are the same in all inertial frames.

Postulate 2: The speed of light in vacuum is constant in all inertial frames, independent of the motion of the source or the observer.

These assumptions challenge the classical Galilean notion of absolute space and time. One is forced to abandon the idea that time is universal—different observers measure different time intervals for the same event.

1.2 Lorentz Transformations

Consider two frames: S and S', where S' moves with velocity v relative to S in the x-direction. The Galilean transformations:

$$x' = x - vt, \quad t' = t$$

fail to preserve the constancy of the speed of light. Instead, the correct transformations are:

$$x' = \gamma(x - vt) \tag{1.1}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{1.2}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor. These are the `Lorentz transformations` and they imply effects such as time dilation and length contraction.

1.3 Spacetime and Simultaneity

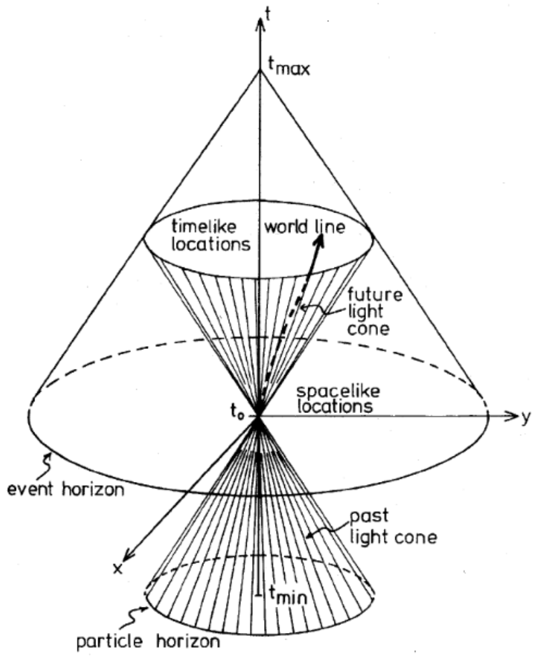

Figure 1.1: *Worldlines and light cones in Minkowski space*

Special Relativity unites space and time into a single entity called **spacetime**. Events are represented as points in this space, and particles trace out worldlines. Two events considered simultaneous in one frame may not be simultaneous in another—a phenomenon known as the **relativity of simultaneity**. This has profound implications, e.g., the famous light-clock thought experiments and the train-lightning paradox.

1.4 Time Dilation and Length Contraction

Time Dilation: A clock moving with speed v relative to an observer ticks slower:

$$\Delta t = \gamma \Delta t_0$$

Length Contraction: An object moving at speed v appears shortened in the direction of motion:

$$L = \frac{L_0}{\gamma}$$

These effects have been experimentally confirmed using fast-moving muons and atomic clocks on airplanes.

1.5 The Twin Paradox

A classic example illustrating time dilation is the `Twin Paradox`. One twin travels at relativistic speeds to a distant star and returns. Upon return, they are younger than the twin who stayed on Earth. This is not a paradox when we account for the fact that the traveling twin undergoes acceleration, making their frame non-inertial during turnaround.

1.6 Relativistic Kinematics

The velocity addition rule in Special Relativity is given by:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

This ensures that no matter how velocities are combined, the resultant speed never exceeds c. The motion of particles must lie within the light cone of space-time diagrams to preserve causality.

1.7 4-Vectors and Invariant Interval

A key development is the use of 4-vectors. The spacetime interval between two events is invariant under Lorentz transformations:

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

This quantity is analogous to distance in Euclidean space but has a mixed signature. It distinguishes between spacelike, timelike, and lightlike separations.

1.8 Geometry Of SpaceTime

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality

1.9 Proper Time

The concept of `Proper Time` τ is central to understanding particle motion in Special Relativity. It is defined as the time measured by a clock moving with the particle itself, making it an **invariant quantity** across all inertial frames.

Along a worldline, the infinitesimal interval is given by the invariant spacetime separation:

$$d\tau = \sqrt{dt^2 - \frac{dx^2}{c^2}} = dt\sqrt{1 - \frac{u^2}{c^2}} = \frac{dt}{\gamma}$$

Here, $u = \frac{dx}{dt}$ is the particle's velocity and $\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$ is the Lorentz factor.

The total proper time experienced by a particle between events is:

$$\tau = \int \frac{dt}{\gamma}$$

1.10 4-Velocity

We now parameterize a particle's spacetime trajectory with proper time τ, writing its position as a 4-vector:

$$X^{\mu}(\tau) = \begin{pmatrix} ct(\tau) \\ x(\tau) \end{pmatrix}$$

Differentiating with respect to τ, we define the **4-velocity** as:

$$U^{\mu} = \frac{dX^{\mu}}{d\tau} = \gamma \begin{pmatrix} c \\ \vec{u} \end{pmatrix}$$

The 4-velocity satisfies the Lorentz-invariant condition:

$$U^{\mu} U_{\mu} = c^2$$

Unlike the 3-velocity, the 4-velocity transforms linearly under Lorentz transformations and is always a *timelike* vector for massive particles.

1.11 4-Momentum

The natural extension of momentum to relativity is the `4-momentum`:

$$P^{\mu} = mU^{\mu} = \begin{pmatrix} \gamma mc \\ \gamma m\vec{u} \end{pmatrix}$$

Its components are: - Time-like component: $P^0 = \gamma mc = \frac{E}{c}$ - Space-like components: $\vec{p} = \gamma m\vec{u}$

The energy-momentum relation follows:

$$P^{\mu} P_{\mu} = E^2/c^2 - p^2 = m^2c^2 \Rightarrow E^2 = p^2c^2 + m^2c^4$$

which generalizes the iconic $E = mc^2$ for particles with kinetic energy.

1.12 Massless Particles

For particles with $m = 0$, such as photons, the concept of proper time breaks down, since their trajectories are `null`—they lie on the light cone. Their 4-momentum satisfies:

$$P^{\mu} P_{\mu} = 0$$

This implies:

$$E = pc \quad \text{and} \quad P^{\mu} = \frac{E}{c} \begin{pmatrix} 1 \\ \hat{p} \end{pmatrix}$$

where $\hat{p}$ is the unit vector in the direction of motion.

These particles must always travel at speed c. The constancy of this speed across all inertial frames is a fundamental postulate of Special Relativity.

An intriguing aspect of massless particles is their transformation under Lorentz boosts. Due to the Doppler effect, their frequency and energy change for different observers, but their speed remains invariant.

1.13 Relativistic Particle Physics

In high-energy regimes, the framework of Special Relativity becomes essential in understanding the behavior and interactions of particles. Processes such as collisions and decays are most naturally analyzed using **4-momentum conservation**, which encompasses both energy and momentum conservation.

1.14 Relativistic Particle Decay

Consider a particle of rest mass m_1 decaying into two particles of masses m_2 and m_3. In the rest frame of the original particle, energy conservation gives:

$$E_1 = m_1 c^2 = \sqrt{p^2c^2 + m_2^2c^4} + \sqrt{p^2c^2 + m_3^2c^4}$$

Solving this provides the momenta and energies of the decay products. For example, in the decay of the Higgs boson $h \to \gamma\gamma$, the photons are emitted back-to-back in the rest frame, each carrying half the total energy:

$$E_\gamma = \frac{1}{2} m_h c^2$$

1.15 Relativistic Collisions

For elastic collisions, where two identical particles collide and scatter, one often analyzes the problem in the `center-of-mass frame`. Suppose both particles have mass m and initial 4-momenta:

$$P_1^\mu = (mc\gamma, mv\gamma, 0, 0), \quad P_2^\mu = (mc\gamma, -mv\gamma, 0, 0)$$

After scattering at an angle θ, the momenta change but total 4-momentum remains conserved:

$$P_1 + P_2 = P_3 + P_4$$

In the lab frame, if one particle is initially at rest, the velocity transformation becomes essential:

$$u = \frac{2v}{1 + v^2/c^2}$$

1.16 Particle Creation and Threshold Energy

In some collisions, new particles are produced, converting kinetic energy into rest mass. Consider the production of a third particle of mass M in addition to two particles of mass m. In the center-of-mass frame, the total energy before and after must satisfy:

$$4m^2\gamma_v^2c^2 = (2mc^2 + Mc^2)^2/c^2$$

Solving for γ_v, we obtain the minimum Lorentz factor and hence the threshold energy per particle required:

$$\gamma_v \geq 1 + \frac{M}{2m}$$

In the lab frame (one particle at rest), the threshold kinetic energy is significantly higher:

$$T \approx \frac{M^2c^2}{2m}$$

This quadratic scaling with M is why modern particle colliders use **colliding beams** rather than a stationary target.

1.17 Compton Scattering

A photon scattering off an electron results in a change in the photon's wavelength—a phenomenon called `Compton Scattering`. Conservation of 4-momentum gives:

$$E'^2 = (E + E_e)^2 - |\vec{p}_\gamma + \vec{p}_e|^2c^2$$

In the electron's rest frame, the Compton formula can be derived:

$$\lambda' - \lambda = \frac{h}{m_ec}(1 - \cos\theta)$$

This effect provides direct experimental confirmation of relativistic energy and momentum conservation in quantum processes.

1.18 Mass-Energy Conversion and Relativity

Special relativity generalizes the classical notion of mass conservation. The relation:

$$E^2 = p^2c^2 + m^2c^4$$

demonstrates that mass and kinetic energy are interchangeable forms of energy. For massive particles at rest:

$$E = mc^2$$

In particle physics, mass can be created from energy (as in pair production), or vice versa (as in annihilation), which forms the foundational principle behind nuclear and high-energy physics.

1.19 Conclusion

Special Relativity fundamentally reshapes our understanding of space, time, and motion. It removes the notion of absolute simultaneity and establishes the constancy of the speed of light as a cornerstone of physical law. The concepts developed here lay the groundwork for General Relativity, where spacetime is no longer flat but curved by mass and energy.

In the next chapter, we explore how these ideas extend to gravity, culminating in the elegant and powerful framework of Einstein's General Relativity.

2. Geodesics In SpaceTime

In Newtonian gravity, particles move under the influence of a force determined by the gravitational field. In Einstein's general relativity, gravity is no longer a force: it is encoded in the geometry of spacetime itself. Particles follow the natural straight-line paths—geodesics—in this curved spacetime. This chapter constructs the equations governing such motion, beginning with the principle of least action in curved space and extending it to relativistic particles.

2.1 The Principle of Least Action

The motion of particles can be derived from a single principle: they follow the path that extremizes an action functional. Consider a particle moving along a path $x^i(t)$ from t_1 to t_2. The action is defined as:

$$S[x^i(t)] = \int_{t_1}^{t_2} dt\, L(x^i, \dot{x}^i)$$

Stationarity of the action $\delta S = 0$ under arbitrary variations yields the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0$$

2.2 Geodesics in Curved Space

In a curved space, distances are measured using a position-dependent metric $g_{ij}(x)$:

$$ds^2 = g_{ij}(x)\, dx^i dx^j$$

The kinetic Lagrangian for a particle of mass m is:

$$L = \frac{m}{2} g_{ij}(x)\, \dot{x}^i \dot{x}^j$$

Solving the Euler-Lagrange equations gives the geodesic equation:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

with Christoffel symbols defined as:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right)$$

Examples

Flat Space in Polar Coordinates: Even flat $\mathbb{R}^3$ space has non-zero Christoffel symbols in curvilinear coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Motion on the Sphere S^2: By fixing $r = R$, the metric becomes:

$$ds^2 = R^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

Geodesics on S^2 are great circles.

2.3 Relativistic Particles

To describe relativistic motion, time and space must be treated on equal footing.

Minkowski Spacetime

The flat spacetime interval is:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

For a particle moving in spacetime, we introduce an arbitrary parameter σ along the worldline, and define the action:

$$S = -mc \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}$$

This action is reparameterization invariant.

2.4 Momentum and Constraints

The canonical momentum is:

$$p^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m u^\mu = m \frac{dx^\mu}{d\tau}$$

which leads to the mass-shell condition:

$$\eta_{\mu\nu} p^\mu p^\nu = -m^2 c^2$$

2.5 Reparameterization Invariance

Since σ is arbitrary, one can choose $\sigma = \tau$ (proper time), simplifying the Lagrangian to:

$$L = -mc^2\sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

2.6 Rediscovering the Forces of Nature

Two important modifications to the action:

- Electromagnetic interaction:

$$S \to S - q\int A_\mu dx^\mu$$

- Gravitational interaction:

$$\eta_{\mu\nu} \to g_{\mu\nu}(x)$$

This naturally incorporates the equivalence principle.

2.7 The Equivalence Principle

All objects fall the same way in a gravitational field. Einstein elevated this to say that gravity and acceleration are locally indistinguishable.

Accelerated Observers

An observer with constant acceleration a follows a hyperbola:

$$x(\tau) = \frac{c^2}{a}\cosh\left(\frac{a\tau}{c}\right), \quad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

In Rindler coordinates, the Minkowski metric becomes:

$$ds^2 = -\left(1 + \frac{a\rho}{c^2}\right)^2 c^2 d\tau^2 + d\rho^2 + dy^2 + dz^2$$

Tidal Forces

While gravity can be transformed away locally, tidal forces—second derivatives of the potential—remain. These manifest as deviations in nearby geodesics.

2.8 Gravitational Time Dilation

The time component of the metric is:

$$g_{00}(r) = -\left(1 + \frac{2\Phi(r)}{c^2}\right)$$

Thus, proper time is:

$$d\tau = \sqrt{1 + \frac{2\Phi(r)}{c^2}}\, dt$$

This leads to gravitational redshift:

$$\omega_A = \omega_B \left(1 + \frac{\Phi(r_A) - \Phi(r_B)}{c^2}\right)$$

2.9 Geodesics in Spacetime

Returning to the relativistic action:

$$S = -mc \int d\sigma \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}$$

the geodesic equation becomes:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

Alternatively, using the action:

$$S = \int d\tau\, g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

gives the same result, provided we impose the constraint:

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2 \quad \text{(timelike)} \quad \text{or} \quad = 0 \quad \text{(null)}$$

This summarizes the elegant unification of motion and geometry: particles traverse geodesics determined by spacetime curvature.

2.10 A First Look at the Schwarzschild Metric

To explore planetary motion and light bending in general relativity, we require the spacetime geometry produced by a spherically symmetric mass M. The Schwarzschild solution, derived later, is:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Far from the mass ($r \to \infty$), this reduces to the flat Minkowski metric. The term $R_s = \frac{2GM}{c^2}$ is called the Schwarzschild radius. For $r = R_s$, the metric appears singular (more on this in later chapters).

2.11 The Geodesic Equations

We compute geodesics using the action:

$$S = \int d\tau \left[-A(r)c^2\dot{t}^2 + A^{-1}(r)\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \right]$$

with $A(r) = 1 - \frac{R_s}{r}$. By symmetry, motion is confined to the equatorial plane ($\theta = \frac{\pi}{2}$).

Two cyclic coordinates yield conserved quantities:

- **Energy:** $E = A(r)c^2\dot{t}$
- **Angular Momentum:** $l = r^2\dot{\phi}$

These reduce the dynamics to an effective potential problem in the radial coordinate.

2.12 Planetary Orbits in Newtonian Mechanics

In Newtonian gravity, planetary motion is governed by:

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{l^2}$$

with $u = \frac{1}{r}$. The solution is:

$$u(\phi) = \frac{GM}{l^2}(1 + e\cos\phi)$$

which describes an elliptical orbit.

2.13 Planetary Orbits in General Relativity

In general relativity, the equation for $u = 1/r$ becomes:

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{l^2} + \frac{3GM}{c^2}u^2$$

The additional $\frac{3GM}{c^2}u^2$ term leads to precession of the perihelion. The total precession per orbit is:

$$\Delta\phi = \frac{6\pi GM}{a(1 - e^2)c^2}$$

This agrees with the observed 43 arcseconds per century for Mercury, a historic success of general relativity.

2.14 The Pull of Other Planets

In practice, the motion of Mercury is also affected by other planets (especially Jupiter). Accounting for these effects is crucial to isolate the relativistic correction. Once done, the residual precession matches GR predictions remarkably well.

2.15 Light Bending

Photons follow null geodesics in the Schwarzschild spacetime. We define the impact parameter $b = \frac{l}{E/c}$, and the deflection angle is calculated perturbatively:

$$\delta\phi = \frac{4GM}{bc^2}$$

Newtonian gravity predicts only half this amount, due to ignoring spacetime curvature's effect on spatial paths.

This prediction was famously confirmed during the 1919 solar eclipse, establishing GR's empirical validity.

Comparison with Newtonian Prediction

The Newtonian result accounts only for the g_{00} term in the metric. General relativity includes spatial curvature (via g_{rr}) as well, doubling the deflection.

2.16 Closing Remarks on Geodesics

The geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

successfully recovers all relativistic corrections to motion. Its elegance lies in its geometric nature: motion is entirely determined by the curvature of spacetime. All test particles, massive or massless, follow these paths dictated by the Einstein equivalence principle.

3. Differential Geometry

To understand general relativity, one must understand the geometric structure of spacetime. This chapter introduces the essential concepts of differential geometry—manifolds, tangent vectors, vector fields, and derivatives—needed to describe curved spacetimes. We do not aim for rigor, but for a logically coherent framework suited for physical intuition.

3.1 Manifolds

A **manifold** is an n-dimensional space that locally looks like $\mathbb{R}^n$ but can have non-trivial global properties.

Topological Spaces

A topological space M is a set with a topology $\mathcal{T}$: a collection of open subsets satisfying:

- $\emptyset, M \in \mathcal{T}$
- Finite intersections of open sets are open.
- Arbitrary unions of open sets are open.

The Hausdorff condition ensures that any two points have disjoint neighborhoods.

Differentiable Manifolds

A manifold is covered by coordinate charts (U_α, ϕ_α) where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ are smooth. Compatibility requires that transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ are smooth.

Examples:

- $\mathbb{R}^n$: trivial manifold
- S^n: requires at least two charts
- $T^2 = S^1 \times S^1$: global topology differs from $\mathbb{R}^2$

Maps Between Manifolds

Smooth maps $f : M \to N$ pull back functions and push forward derivatives. These become crucial when defining vector fields and tensors.

3.2 Tangent Spaces

At each point $p \in M$, the **tangent space** $T_p(M)$ contains all possible tangent vectors at p.

Tangent Vectors

We define vectors at p as directional derivatives acting on functions:

$$V(f) = \frac{d}{d\lambda} f(x^\mu(\lambda))\bigg|_{\lambda=0}$$

Given coordinates x^μ, basis vectors are $\left\{\frac{\partial}{\partial x^\mu}\right\}$.

Vector Fields

A vector field assigns to each point $p \in M$ a vector $V \in T_p(M)$. In coordinates:

$$V = V^\mu(x) \frac{\partial}{\partial x^\mu}$$

They obey the Leibniz rule when acting on functions.

Integral Curves

Given a vector field V, an integral curve $x^\mu(\lambda)$ satisfies:

$$\frac{dx^\mu}{d\lambda} = V^\mu(x)$$

These curves represent "flow lines" of the field.

The Lie Derivative

Given a vector field X, the Lie derivative of another field Y measures the change of Y along the flow of X:

$$\mathcal{L}_X Y = [X, Y]$$

This is the commutator of vector fields. It encodes how one field "drags" another along.

3.3 Useful Notes

- Tangent vectors form a vector space.
- Coordinate transformations change the basis of $T_p(M)$.
- The Lie derivative is a coordinate-free concept.
- Not all vector fields generate diffeomorphisms (invertible maps).

3.4 Tensors

We now define tensors, the primary objects that inhabit our manifolds and encode physical information.

Covectors and One-Forms

At a point $p \in M$, the dual space to the tangent space $T_p(M)$ is the **cotangent space** $T_p^*(M)$. Elements of this space are **covectors**, or **one-forms**.

Given a coordinate basis $\{\partial/\partial x^\mu\}$ for $T_p(M)$, the dual basis for $T_p^*(M)$ is denoted $\{dx^\mu\}$, satisfying:

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu$$

Any one-form ω can be written as:

$$\omega = \omega_\mu dx^\mu$$

The exterior derivative of a function f yields the one-form:

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu$$

The Lie Derivative Revisited

The Lie derivative $\mathcal{L}_X \omega$ of a one-form ω along a vector field X is defined as:

$$(\mathcal{L}_X \omega)(Y) = X(\omega(Y)) - \omega([X, Y])$$

This encodes how differential forms change along flows generated by vector fields.

Tensors and Tensor Fields

A tensor at p is a multilinear map that takes k vectors and l one-forms:

$$T : \underbrace{T_p^*(M) \times \cdots \times T_p^*(M)}_{k} \times \underbrace{T_p(M) \times \cdots \times T_p(M)}_{l} \to \mathbb{R}$$

Tensors transform under coordinate changes via appropriate Jacobians. A **tensor field** assigns a tensor to each point of the manifold smoothly.

3.5 Differential Forms

Differential forms are totally antisymmetric tensors of type $(0, p)$, with deep applications in geometry and physics.

The Exterior Derivative

The exterior derivative $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$ satisfies:

- Linearity: $d(\omega + \eta) = d\omega + d\eta$
- Leibniz Rule: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$
- Nilpotency: $d^2 = 0$

Forms You Know and Love

Familiar vector calculus operations are recast:

- Gradient: df
- Curl: $d\omega$ for 1-forms
- Divergence: $*d*\omega$, using the Hodge star

A Sniff of de Rham Cohomology

Since $d^2 = 0$, we define:

$$Z^p(M) = \ker(d : \Lambda^p \to \Lambda^{p+1}), \quad B^p(M) = \mathrm{im}(d : \Lambda^{p-1} \to \Lambda^p)$$

The de Rham cohomology is $H^p(M) = Z^p(M)/B^p(M)$, which classifies closed forms modulo exact ones.

Integration

A p-form can be integrated over a p-dimensional manifold. Integration is coordinate-independent and respects orientation. For example, for a 2-form ω on a 2D surface:

$$\int_M \omega = \int_M \omega_{12} dx^1 \wedge dx^2$$

Stokes' Theorem

The grand unification of various integral theorems:

$$\int_M d\omega = \int_{\partial M} \omega$$

Examples:

- For 0-forms: Fundamental Theorem of Calculus

- For 1-forms: Green's Theorem
- For 2-forms: Gauss' Divergence Theorem

This single theorem encompasses much of classical vector calculus and is a cornerstone of modern differential geometry.

4. Riemannian Geometry

To describe gravity as geometry, we introduce a key player: the **metric**, a symmetric, non-degenerate (0,2)-tensor field that allows us to measure distances, angles, and curvature on a manifold. This section develops the formalism of Riemannian and Lorentzian geometry, including covariant derivatives, curvature, and the role of the metric in field theory.

4.1 The Metric

The metric g assigns an inner product on the tangent space $T_p(M)$ at every point $p \in M$:

$$g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$$

or, in shorthand, $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$

Riemannian Manifolds

These have positive-definite metrics (e.g., Euclidean space):

$$g = \delta_{\mu\nu}dx^\mu \otimes dx^\nu$$

$$|X| = \sqrt{g(X,X)} \quad \text{and} \quad g(X,Y) = |X||Y|\cos\theta$$

Lorentzian Manifolds

Here, one diagonal component is negative (e.g., Minkowski space). The signature is typically $(-+++)$. Lightcones arise from the metric structure and define causal structure.

The Joys of a Metric

- Defines lengths, angles, and volumes.
- Provides isomorphism between vectors and covectors.
- Enables raising/lowering of indices via $X_\mu = g_{\mu\nu}X^\nu$

The Volume Form

For Riemannian manifolds:

$$v = \sqrt{\det g_{\mu\nu}}\, dx^1 \wedge \cdots \wedge dx^n$$

For Lorentzian manifolds:

$$v = \sqrt{-g}\, dx^0 \wedge \cdots \wedge dx^{n-1}$$

Hodge Star and Inner Products

The Hodge dual maps p-forms to $(n-p)$-forms:

$$(*\omega)_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{p!}\sqrt{|g|}\, \epsilon_{\mu_1 \ldots \mu_{n-p}\nu_1 \ldots \nu_p} \omega^{\nu_1 \ldots \nu_p}$$

This allows defining inner products on $\Lambda^p(M)$:

$$\langle \eta, \omega \rangle = \int_M \eta \wedge *\omega$$

Hodge Theory

The adjoint of the exterior derivative:

$$d^\dagger = \pm(-1)^{np+n+1} * d*$$

The Laplacian:

$$\Delta = dd^\dagger + d^\dagger d$$

Harmonic forms satisfy $\Delta\omega = 0$ and are both closed and co-closed.

4.2 Connections and Curvature

The Covariant Derivative

To differentiate tensor fields on a manifold, we introduce a connection ∇. For vector fields X, Y:

$$\nabla_X Y$$

In coordinates, the covariant derivative becomes:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda$$

Torsion and Curvature

The torsion tensor:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

The Riemann curvature tensor:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

The Levi-Civita Connection
The unique connection that is:

- **Torsion-free**: $T = 0$
- **Metric-compatible**: $\nabla g = 0$

This leads to the Christoffel symbols:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

The Divergence Theorem
If j^μ is a vector field:

$$\int_M \nabla_\mu j^\mu \sqrt{|g|}\, d^n x = \int_{\partial M} j^\mu n_\mu \sqrt{|h|}\, d^{n-1} x$$

The Maxwell Action
An elegant application:

$$S = -\frac{1}{4} \int_M F_{\mu\nu} F^{\mu\nu} \sqrt{|g|}\, d^4 x$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

4.3 Parallel Transport

Parallel transport provides a way to move vectors along curves while keeping them "unchanged" with respect to the connection.

A vector field Y is said to be parallely transported along a curve with tangent X if:

$$\nabla_X Y = 0$$

This leads to:

$$\frac{dY^\mu}{d\tau} + X^\nu \Gamma^\mu_{\nu\rho} Y^\rho = 0$$

These are first-order differential equations solvable given an initial vector.

Path Dependence
Parallel transport is path dependent due to curvature. The change in a vector transported around a loop reveals information about the curvature tensor.

4.4 Geodesics Revisited

A geodesic is a curve whose tangent vector X satisfies:

$$\nabla_X X = 0 \Rightarrow \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

This is the geodesic equation again, now viewed through the lens of parallel transport.

Normal Coordinates

At any point $p \in M$, one can construct coordinates such that:

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \quad \partial_\rho g_{\mu\nu}(p) = 0 \Rightarrow \Gamma^\mu_{\nu\rho}(p) = 0$$

These are called normal coordinates, and locally they make spacetime look flat.

4.5 Geodesic Deviation

To study how nearby geodesics deviate, introduce a deviation vector η^μ. The geodesic deviation equation is:

$$\frac{D^2 \eta^\mu}{D\tau^2} = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho \eta^\sigma$$

This equation plays a central role in gravitational tidal effects.

4.6 More on the Riemann Tensor and Its Friends

The Ricci and Einstein Tensors

The Ricci tensor and scalar:

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}$$

The Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

This tensor satisfies $\nabla^\mu G_{\mu\nu} = 0$ due to the Bianchi identity.

Connection 1-Forms and Curvature 2-Forms

Using a non-coordinate basis $\{\hat{e}_a\}$, define the connection 1-forms:

$$\omega^a{}_b = \Gamma^a_{cb} \hat{\theta}^c$$

and curvature 2-forms:

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

These encapsulate the curvature in elegant differential form language.

An Example: Schwarzschild Metric

In spherical coordinates, the Schwarzschild line element is:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Curvature tensors derived from this metric confirm the central role of $R_{\mu\nu} = 0$ in vacuum solutions.

Relation to Yang-Mills Theory

A formal analogy exists:

$$[\nabla_\mu, \nabla_\nu] V^\sigma = R^\sigma{}_{\rho\mu\nu} V^\rho \quad \leftrightarrow \quad [D_\mu, D_\nu] = F_{\mu\nu}$$

Here, curvature acts like a field strength, and the Christoffel connection parallels a gauge potential.

5. Einstein's Equations

Having laid the geometric foundations of spacetime, we now ask: how does gravity emerge dynamically from this structure? In General Relativity, the metric $g_{\mu\nu}(x)$ is a dynamical field, and the Einstein equations describe its evolution. These are derived from an action principle, culminating in the Einstein-Hilbert action, the cornerstone of the theory.

5.1 The Einstein-Hilbert Action

The simplest action intrinsic to the metric is:

$$S = \int d^4x \sqrt{-g}R \tag{4.1}$$

This action is known as the **Einstein-Hilbert action**. Here, $g = \det g_{\mu\nu}$ and R is the Ricci scalar curvature. The minus sign under the square root arises from the Lorentzian signature of spacetime.

Why It Works

- The Levi-Civita connection depends on the metric: $\Gamma \sim \partial g$ - The curvature depends on the connection: $R \sim \partial\Gamma + \Gamma^2$ - Therefore, the action involves second derivatives of the metric, similar to other second-order field theories.

Varying the Action

To derive the equations of motion, we vary $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The variation yields:

$$\delta S = \int d^4x \left[(\delta\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\right] \tag{4.2}$$

Key identities: - Variation of the inverse metric:

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$$

- Variation of the determinant:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$$

Final Result

After discarding total derivatives and simplifying, the Euler-Lagrange equations from the Einstein-Hilbert action give:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

This is the vacuum Einstein field equation.

5.2 An Aside on Dimensional Analysis

Let us assess the dimensions of Newton's constant G. In natural units ($c = \hbar = 1$), it has dimension:

$$[G] = M^{-2}$$

This defines the reduced Planck mass:

$$M_{\text{pl}}^2 = \frac{1}{8\pi G} \Rightarrow S = \frac{1}{2}M_{\text{pl}}^2 \int d^4x\,\sqrt{-g}R$$

Although it is tempting to set $G = 1$, dimensional analysis is valuable and we retain G in this course.

5.3 The Cosmological Constant

There exists a simpler scalar invariant one can add to the action:

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}(R - 2\Lambda)$$

This is the Einstein-Hilbert action with a cosmological constant Λ. Varying this action leads to the modified field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

or, equivalently (upon contraction):

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

The cosmological constant acts like a vacuum energy density.

6. Black Holes

Black holes are compelling solutions of the Einstein equations, exhibiting unique physical and geometrical properties. This chapter begins by examining the Schwarzschild metric and gradually introduces techniques to understand its horizon and singularity structure.

6.1 The Schwarzschild Solution

We've previously encountered the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{6.1}$$

This is a vacuum solution $R_{\mu\nu} = 0$, describing the spacetime outside a spherical mass M. The metric component g_{00} agrees with the Newtonian potential:

$$\Phi = -\frac{GM}{r}$$

Komar Mass

The Schwarzschild solution admits a timelike Killing vector $K = \partial_t$. The Komar integral gives the mass:

$$M_{\text{Komar}} = -\frac{1}{8\pi G} \int_{S^2} \star dK = M$$

This provides a coordinate-independent way to extract the mass from spacetime geometry.

6.2 Birkhoff's Theorem

This theorem states that any spherically symmetric vacuum solution is necessarily static and asymptotically flat. The Schwarzschild solution is the unique such solution: - No gravitational radiation exists in a spherically symmetric vacuum. - The exterior solution remains unchanged even if the star pulsates.

6.3 A First Look at the Horizon

At $r = 2GM$, the Schwarzschild radius, we encounter coordinate singularities:

$$g_{tt} \to 0, \quad g_{rr} \to \infty$$

However, this is not a true curvature singularity. The Kretschmann scalar remains finite:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}$$

Thus, the singularity at $r = 2GM$ is a coordinate artifact.

6.4 Eddington-Finkelstein Coordinates

To remove the coordinate singularity, introduce the ingoing null coordinate:

$$v = t + r_* \quad \text{where} \quad r_* = r + 2GM \log\left|\frac{r}{2GM} - 1\right|$$

The metric becomes:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega^2$$

This form is regular at $r = 2GM$, extending the manifold across the horizon. Radial null geodesics satisfy:

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)$$

indicating that nothing escapes from $r < 2GM$.

Lightcones at the Horizon

- Outside the horizon ($r > 2GM$), light cones tilt inward. - At the horizon ($r = 2GM$), light cones are tangent to the horizon. - Inside the horizon ($r < 2GM$), all future-directed trajectories move toward smaller r.

`Result:` Once inside, nothing—not even light—can escape.

6.5 Kruskal Spacetime

To uncover the true structure of the Schwarzschild metric and avoid the horizon singularity at $r = 2GM$, we introduce **Kruskal coordinates**.

Kruskal Coordinates

Starting from the tortoise coordinate:

$$r_* = r + 2GM \log\left|\frac{r}{2GM} - 1\right|$$

we define null coordinates:

$$u = t - r_* \qquad v = t + r_*$$

Then, introduce Kruskal-Szekeres coordinates:

$$U = -e^{-u/4GM} \qquad V = e^{v/4GM}$$

The metric becomes:

$$ds^2 = -\frac{32G^3M^3}{r}e^{-r/2GM}dUdV + r^2d\Omega^2$$

This form is regular at the horizon and fully extends the spacetime across $r = 2GM$.

Global Structure

- The singularity at $r = 0$ remains a true curvature singularity. - Kruskal spacetime contains four regions: - Region I: $r > 2GM$, our external universe. - Region II: $r < 2GM$, inside the black hole. - Region III: A white hole region. - Region IV: A second asymptotically flat universe.

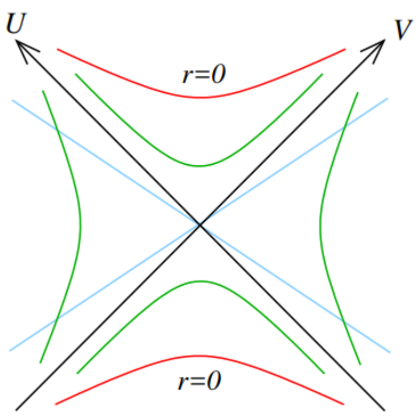

Figure 6.1: Kruskal diagram: light cones are at 45°, making causal structure transparent.

`Note:` No observer in Region I can access Region II or escape from it. Similarly, no information can come out of Region II.

6.6 Forming a Black Hole: Weak Cosmic Censorship

While the eternal Schwarzschild black hole is mathematically elegant, real black holes form from collapsing matter. This motivates the **cosmic censorship conjecture**.

Gravitational Collapse

We study a collapsing dust ball (Oppenheimer-Snyder model): - Interior: described by a closed Friedmann-Robertson-Walker (FRW) universe. - Exterior: Schwarzschild metric.

Matching the metrics across the boundary shows that the star's surface moves inward, eventually crossing the Schwarzschild radius, forming an event horizon.

Penrose Diagrams

To visualize causal structure, Penrose diagrams compactify spacetime. For collapsing stars: - The singularity lies in the future of the horizon. - There is no Region III or IV; those are artifacts of eternal black holes. - This more physical picture is consistent with a universe beginning without a singularity in the past.

Weak Cosmic Censorship Conjecture: All singularities are hidden behind event horizons. No "naked singularities" are visible to distant observers.

Motivation: Ensures determinism in general relativity. Otherwise, the future evolution of spacetime could not be uniquely predicted from initial data.

7. Simulation Theory

Abstract

The Schwarzschild metric is one of the most fundamental solutions to the Einstein Field Equations and represents the spacetime geometry outside a non-rotating, spherically symmetric mass. This project aims to explore and visualise geodesics (the paths of particles and light) in the Schwarzschild spacetime. We numerically integrate the geodesic equations and render the resulting trajectories using animations. Different physical scenarios like photon spheres, perihelion precession, and horizon approaches are examined.

Objective

Our goal is to simulate the geodesic motion of massive and massless particles in Schwarzschild spacetime using numerical methods. The objectives include:

- Understanding and solving geodesic equations in Schwarzschild geometry.
- Simulating trajectories using numerical integration techniques.
- Visualising different physical cases of particle motion with animations.
- Identifying and interpreting key relativistic effects like perihelion precession and the photon sphere.

Introduction

In General Relativity, geodesics represent the paths of free-falling particles in a curved spacetime. The Schwarzschild solution is a static, spherically symmetric vacuum solution to Einstein's equations, described by the metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta, d\phi^2$$

Here, M represents the mass of the black hole, and (t, r, θ, ϕ) are the Schwarzschild coordinates. The event horizon lies at $r = 2M$. In this project, we study test particle dynamics confined to the equatorial plane ($\theta = \pi/2$), which simplifies the analysis without loss of generality.

Geodesic Equations

The motion of particles and light is governed by the geodesic equations derived from the Schwarzschild metric. The Lagrangian for the system in the equatorial plane $\theta = \pi/2$ is:

$$\mathcal{L} = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$$

From the Euler-Lagrange equations, two constants of motion can be identified:

$$E = \left(1 - \frac{2M}{r}\right)\dot{t}\ L \qquad = r^2\dot{\phi}$$

where E is the energy per unit mass and L is the angular momentum per unit mass.

Substituting these into the metric, we obtain the radial equation of motion:

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right)$$

where $\epsilon = 1$ for massive particles and $\epsilon = 0$ for photons. This equation forms the basis of our numerical integration to obtain particle trajectories. section*ODE System for Numerical Integration To numerically solve the geodesic equations, we first convert the second-order differential equations into a system of first-order ordinary differential equations (ODEs). This is necessary as most numerical solvers, including those in standard Python libraries like SciPy, operate on first-order systems.

The Schwarzschild metric in the equatorial plane ($\theta = \pi/2$) leads to the geodesic Lagrangian:

$$\mathcal{L} = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$$

From this Lagrangian, we derive the following conserved quantities:

$$E = \left(1 - \frac{2M}{r}\right)\dot{t}\ L \qquad = r^2\dot{\phi}$$

Substituting into the metric constraint, the radial equation of motion becomes:

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right)$$

where $\epsilon = 1$ for massive particles and $\epsilon = 0$ for photons.

To prepare for numerical integration, we define the state vector $\vec{y} = [t, r, \phi, p_r]$ where $p_r = \dot{r}$. Using this, the first-order system of ODEs is:

$$\frac{dt}{d\lambda} = \frac{E}{1-\frac{2M}{r}} \frac{dr}{d\lambda} = p_r \frac{d\phi}{d\lambda} = \frac{L}{r^2} \frac{dp_r}{d\lambda} = -\frac{M}{r^2}\left(\frac{E^2}{(1-\frac{2M}{r})^2}\right) + \frac{L^2}{r^3} - \frac{M}{r^2}\left(1-\frac{2M}{r}\right)$$

These equations can now be passed to a numerical integrator such as scipy.integrate.solve to obtain trajectories for test particles around a Schwarzschild black hole. section*Conserved Quantities and Initial Conditions In the Schwarzschild spacetime, the conservation laws are derived from the symmetries of the metric. Due to time translational invariance and spherical symmetry, the energy and angular momentum of a test particle are conserved along its geodesic motion.

Energy and Angular Momentum

The conserved energy E and angular momentum L are given by:

$$E = \left(1-\frac{2M}{r}\right)\dot{t}\ L \qquad = r^2\dot{\phi}$$

Here, $\dot{t}$ and $\dot{\phi}$ are derivatives with respect to the affine parameter λ, and M is the Schwarzschild radius (related to the mass of the black hole).

Conserved Quantities and Initial Conditions

From the geodesic equations derived above, we can see that the Schwarzschild spacetime has two constants of motion: the specific energy E and the specific angular momentum L.

The conserved quantities are:

$$E = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda},\ L \qquad = r^2\frac{d\phi}{d\lambda}.$$

These equations can be used to find $\frac{dt}{d\lambda}$ and $\frac{d\phi}{d\lambda}$ in terms of r:

$$\frac{dt}{d\lambda} = \frac{E}{1-\frac{2M}{r}},\ \frac{d\phi}{d\lambda} \qquad = \frac{L}{r^2}.$$

These expressions are substituted into the radial equation and help reduce the order of the ODE system.

To solve the geodesic equations numerically, we need to define initial conditions for the radial position r, the radial velocity $\frac{dr}{d\lambda}$, the coordinate time t, and the azimuthal angle ϕ.

Note that t and ϕ can be set to zero initially without loss of generality due to the symmetry of the Schwarzschild metric. The values of E, L, and initial r and $\frac{dr}{d\lambda}$ are chosen depending on the physical situation to be simulated.

8. Simulations

8.1 Photon Sphere

Parameters Used:

$$E = 1.0, \quad L_z = 3\sqrt{3}, \quad \varepsilon = 0, \quad r_0 = 3.0, \quad \lambda_{\text{max}} = 12$$

Physical Significance:
The photon sphere is a critical surface at $r = 3M$ where massless particles (photons) can orbit a Schwarzschild black hole in unstable circular orbits. Any deviation causes the photon to either escape or fall into the black hole.

Mathematical Context:
The effective potential for a photon is given by:

$$V_{\text{eff}}(r) = \left(1 - \frac{2M}{r}\right)\frac{L_z^2}{r^2}$$

The condition for circular orbits:

$$\frac{dV_{\text{eff}}}{dr} = 0 \quad \Rightarrow \quad r = 3M$$

Observation from Simulation:
The simulation shows a photon circling the black hole at $r = 3M$, tracing a perfect circular trajectory as expected.

8.2 Perihelion Precession

Parameters Used:

$$E = 0.97, \quad L_z = 7.0, \quad \varepsilon = -1, \quad r_0 = 20.0, \quad \lambda_{\text{max}} = 1000$$

Physical Significance:
The orbit of a massive particle is not closed due to spacetime curvature, leading to the precession of the perihelion with each revolution — famously observed in Mercury's orbit.

Mathematical Context:
The effective potential is:

$$V_{\text{eff}}(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L_z^2}{r^2}\right)$$

The particle oscillates between r_{min} and r_{max}, with:

$$E^2 < 1 \quad \text{(Bound Orbit)}$$

Observation from Simulation:
The particle traces a rosette-like orbit, showing the classic relativistic precession of the perihelion.

8.3 Event Horizon Approach

Parameters Used:

$$E = 1.0, \quad L_z = 0.0, \quad \varepsilon = -1, \quad r_0 = 10.0, \quad \lambda_{\text{max}} = 250$$

Physical Significance:
With zero angular momentum, the massive particle undergoes pure radial infall into the black hole.
Mathematical Context:
The radial equation:

$$\frac{dr}{d\tau} = -\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)}$$

At $r = 2M$, the particle reaches the event horizon.
Observation from Simulation:
The particle falls straight inward, disappearing at the horizon — in line with GR predictions.

8.4 Whirlzoom Motion

Parameters Used:

$$E = 0.995, \quad L_z = 5.0, \quad \varepsilon = -1, \quad r_0 = 8.0, \quad \lambda_{\text{max}} = 500$$

Physical Significance:
A near-critical orbit in which a massive particle executes multiple whirls near the black hole before escaping. This motion is relevant in EMRIs.
Mathematical Context:
Near the ISCO (innermost stable circular orbit), the particle enters a region of steep potential:

$$V_{\text{eff}}(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L_z^2}{r^2}\right)$$

Small perturbations yield whirl-zoom dynamics.

Observation from Simulation:
The trajectory starts far, zooms in to execute several tight orbits, and finally zooms out.

8.5 PolarPS Case

The PolarPS case refers to the simulation of a photon orbiting at the photon sphere radius in Schwarzschild spacetime. The photon sphere is a spherical region where gravity is strong enough that photons can travel in circular orbits. This occurs at a radial coordinate:

$$r = 3M$$

where M is the mass of the central black hole.

This case demonstrates an unstable circular photon orbit. Any small perturbation would cause the photon to either escape to infinity or fall into the black hole. To simulate this, we initialise the geodesic equations with the following parameters:

$$M = 1 \quad r_0 = 3M = 3 \quad \left.\frac{dr}{d\lambda}\right|_{\lambda=0} = 0 \quad L = 3.464 \quad E = L/r_0 = 1.154$$

These values ensure that the photon starts at $r = 3M$ and stays in a nearly circular orbit, tracing the photon sphere.

8.6 PolarPeri Case

The PolarPEri case models the perihelion precession of a massive test particle. In Newtonian gravity, elliptical orbits are closed, but in General Relativity, the presence of spacetime curvature causes the perihelion to advance with each orbit. This effect is especially noticeable near strong gravitational sources.

In this simulation, the following parameters are used:

$$M = 1 \quad r_0 = 7 \quad \left.\frac{dr}{d\lambda}\right|_{\lambda=0} = 0 \quad L = 4 \quad E = 0.943$$

These values result in an elliptical-like orbit around the black hole with a clearly observable precession of the perihelion over time. The test particle does not maintain a fixed elliptical orbit but rotates slowly in the azimuthal direction, forming a rosette pattern.

8.7 Code

The relevant code is available on the following link: https://drive.google.com/file/d/1DBqBbUvOIQLz9si5t7aumEDrv4qPp-x5/view?usp=sharing

Overview

This section explains the Python code written to simulate geodesic trajectories around a Schwarzschild black hole using `Manim`. The simulation solves the geodesic equations and visualizes the result with aesthetic animations.

Core Libraries

The implementation uses several Python libraries:

- **NumPy:** for numerical operations.
- **SciPy:** to integrate differential equations with `solveivp`.
- **Pandas:** to store trajectory data.
- **Manim:** to animate the simulation.

Geodesic Simulator Class

This class encapsulates the physics:

- Initialized with parameters: energy E, angular momentum L_z, rest mass ε, and initial radius r_0.
- Encodes Schwarzschild geodesic equations as an ODE system.
- Uses `solveivp` to compute the trajectory in terms of affine parameter λ.
- Stores the result as a Pandas DataFrame for easy access.

Scene Setup and Logic

For each case, a subclass of `BaseSimulationScene` is used. This manages all rendering:

- The scene has a white background to enhance contrast.
- Concentric circles and radial lines mimic Schwarzschild coordinates.
- A central black disk represents the event horizon at $r = 2M$.
- A green dot shows the test particle. `TracedPath` leaves behind its trail.
- A timer using `ValueTracker` and `DecimalNumber` displays evolving λ.

Animation Process

1. A dictionary of preset parameters selects the simulation case.
2. The geodesic is computed with the simulator.
3. Polar coordinates are converted to Cartesian using $x = r\cos\phi$, $y = r\sin\phi$.
4. `interp1d` is used to allow smooth animation based on these coordinates.

Design Considerations

The simulation emphasizes modularity and clarity:

- **Reusable:** Code components are reusable across different physical scenarios.
- **Readable:** Physics and rendering are cleanly separated.
- **Accurate:** Based on rigorous geodesic equations from General Relativity.
- **Informative:** Designed for both educational and scientific use.

8.8 Simulation Videos

The relevant Simulation Videos are available at the following links:
Proton Sphere: https://drive.google.com/file/d/1tighmKrIB7FX2o_74WulCj5FeueAPXHH/view?usp=sharing
Perihelion Precession: https://drive.google.com/file/d/1tighmKrIB7FX2o_74WulCj5FeueAPX
view?usp=sharing
Event Horizon Approach: https://drive.google.com/file/d/1hk7OcyTpyxjgnvB334Hnx1ibvsCQ
view?usp=sharing
WhirlZoom Motion: https://drive.google.com/file/d/1K5NGyqelhLEvbMHc2TG-MvMNGK786YaC/view?usp=sharing
PolarPS Motion: https://drive.google.com/file/d/1VG3rgXzlzRcasEneNWNFd982p8KdSvGP/view?usp=sharing
Polarperi Motion: https://drive.google.com/file/d/17tAgSQIuDNjuyRx0Uh-1e8-DQQJ9vFM7/view?usp=sharing

9. Bibliography

The information used to prepare this report has been sourced from the material prepared by Professor David Tong, Department Of Applied Mathematics and Theoretical Physics, Centre For Mathematical Sciences, University Of Cambridge.