KSP ENDTERM REPORT

A Comprehensive Report

Special and General Relativity along with Simulations

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Contents

1 Special Relativity

1.1 Need for Relativity: Failure of Newtonian Mechanics at Extreme Conditions

Classical Newtonian mechanics, while highly successful in describing everyday phenomena, fails under extreme conditions:

- **High Velocities (Approaching the Speed of Light)**
  - Newton's laws assume absolute time and space, independent of the observer's motion.
  - At speeds approaching the speed of light ($c \approx 3 \times 10^8$ m/s), Galilean relativity breaks down.
  - Maxwell's equations predict electromagnetic waves travel at speed c, conflicting with Newtonian mechanics.
  - The Michelson-Morley experiment (1887) showed no detectable "aether wind," contradicting the Newtonian notion of a preferred reference frame.
- **Strong Gravitational Fields**
  - Newtonian gravity assumes instantaneous force propagation, incompatible with relativity.
  - General Relativity later resolved this by describing gravity as spacetime curvature.

1.2 Postulates of Special Relativity

Einstein resolved these issues with two postulates:

1. **Principle of Relativity**: The laws of physics are the same in all inertial reference frames.
2. **Invariance of the Speed of Light**: The speed of light in vacuum (c) is constant and independent of the motion of the source or observer.

1.3 Lorentz Transformations

The Lorentz transformations relate space and time coordinates between two inertial frames moving at relative velocity v along the x-axis:

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.

These transformations preserve the spacetime interval $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$ and ensure c remains constant in all frames.

1.4 Minkowski Space and Spacetime Diagrams

Special relativity unifies space and time into a single four-dimensional structure called **spacetime**. This was formalized by Hermann Minkowski, who showed that time should be treated as a fourth coordinate.

In Minkowski space:

- Events are represented as points with coordinates (ct, x, y, z).
- The spacetime interval between two events is given by:

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

- This interval is invariant under Lorentz transformations — all observers agree on its value.

Based on the sign of s^2, the interval is classified as:

- **Timelike** ($s^2 > 0$): Events can influence each other causally.
- **Lightlike** or **null** ($s^2 = 0$): Events connected by light signals.
- **Spacelike** ($s^2 < 0$): No causal connection; information cannot travel fast enough.

Spacetime diagrams are 2D plots (typically ct vs. x) that visually represent events, worldlines of particles, and the structure of light cones:

- The line $x = ct$ and $x = -ct$ form the boundaries of the light cone.
- Worldlines inside the light cone represent possible paths of particles traveling slower than light.
- Worldlines outside the cone would require faster-than-light motion, which is forbidden.

Minkowski space makes the geometry of special relativity intuitive and helps visualize phenomena like simultaneity and time dilation.

1.5 Causality and the Relativity of Simultaneity

Causality is the principle that a cause must precede its effect. In special relativity, preserving causality means that no information or influence can travel faster than the speed of light.

Events that are **timelike** separated can be causally connected — one can influence the other — because there is enough time for a light signal (or slower) to travel between them. Events that are **spacelike** separated cannot be causally connected since doing so would require faster-than-light transmission, violating special relativity.

Relativity of simultaneity is the idea that simultaneity is not absolute. Two events that are simultaneous in one inertial frame may not be simultaneous in another. This arises directly from the Lorentz transformations.

- Suppose two events occur at the same time t but different positions x_1 and x_2 in one frame.

- In another frame moving at velocity v, the time difference becomes:

$$\Delta t' = \gamma \left(\Delta t - \frac{v \Delta x}{c^2} \right)$$

- If $\Delta t = 0$, then $\Delta t' = -\gamma \frac{v\Delta x}{c^2} \neq 0$, meaning the events are no longer simultaneous.

This has deep implications:

- There is no universal "now" that all observers agree on.
- Time ordering of spacelike-separated events depends on the observer's frame of reference.

Despite this, causality is always preserved — if one event can influence another, all observers will agree on which came first.

1.6 Time Dilation and Length Contraction

Special relativity predicts that time and length are not absolute — they depend on the motion of the observer. Two key consequences are:

Time Dilation

A moving clock ticks more slowly compared to one at rest. If a time interval Δt_0 is measured in the clock's own rest frame (called proper time), then an observer moving at speed v relative to the clock measures:

$$\Delta t = \gamma \Delta t_0, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This effect has been experimentally confirmed, such as by observing muons created in the upper atmosphere surviving longer than expected due to their high speeds.

Length Contraction

Objects moving relative to an observer appear shorter along the direction of motion. If L_0 is the proper length (measured in the object's rest frame), then the length measured by an observer for whom the object is moving at speed v is:

$$L = \frac{L_0}{\gamma}$$

Only lengths along the direction of motion are affected; transverse dimensions remain unchanged. This effect becomes significant at speeds close to the speed of light.

Both phenomena arise directly from the Lorentz transformations and are not illusions — they reflect the geometry of spacetime itself.

1.7 Invariant Interval and Lorentz Group

In special relativity, the concept of distance is generalized to four-dimensional spacetime through the **invariant interval**. For two events with coordinates (t, x, y, z) and (t', x', y', z'), the spacetime interval s^2 is defined as:

$$s^2 = c^2(t' - t)^2 - (x' - x)^2 - (y' - y)^2 - (z' - z)^2$$

This interval is invariant under Lorentz transformations, meaning all inertial observers agree on its value. It replaces the idea of absolute time and space with a unified spacetime structure.

- If $s^2 > 0$, the interval is **timelike** — events can be causally connected.
- If $s^2 = 0$, the interval is **lightlike** — events are connected by a light signal.
- If $s^2 < 0$, the interval is **spacelike** — no causal influence is possible.

The Lorentz Group

The set of all transformations that preserve the invariant interval forms the **Lorentz group**, denoted $O(1, 3)$. These include:

- Boosts (transformations between frames moving at constant velocity)
- Rotations in 3D space
- Reflections and time reversal (in the full group)

The subgroup that preserves orientation and the direction of time is called the **proper orthochronous Lorentz group**, denoted $SO^+(1, 3)$. Lorentz transformations can be represented by matrices Λ satisfying:

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \, \eta_{\mu\nu} = \eta_{\alpha\beta}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric.

This formalism ensures that the laws of physics — especially Maxwell's equations and the spacetime structure — remain form-invariant under all inertial frames.

1.8 Proper Time

Proper time τ is the time measured by a clock that moves along with the particle — i.e., in the particle's rest frame. It is an invariant quantity and plays a fundamental role in relativistic dynamics.

For an infinitesimal displacement in spacetime:

$$d\tau = \frac{1}{c}\sqrt{ds^2} = \sqrt{dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)}$$

If the particle moves with velocity v, then:

$$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma}$$

This makes proper time the natural parameter along a particle's worldline, analogous to arc length in geometry.

1.9 4-Velocity

The **4-velocity** U^μ is the rate of change of the spacetime position x^μ with respect to proper time τ:

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v})$$

Here, $\vec{v}$ is the 3-velocity and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

Properties:

- It transforms as a 4-vector under Lorentz transformations.
- Its Minkowski norm is invariant:

$$U^\mu U_\mu = c^2$$

1.10 4-Momentum

The **4-momentum** P^μ is defined as:

$$P^\mu = mU^\mu = (\gamma mc, \gamma m\vec{v})$$

This generalizes the classical momentum and energy into a single 4-vector. Its components are:

- $P^0 = \gamma mc \to$ total energy divided by c: $E = \gamma mc^2$
- $\vec{P} = \gamma m\vec{v} \to$ relativistic 3-momentum

The Minkowski norm of 4-momentum gives the rest mass:

$$P^\mu P_\mu = E^2/c^2 - |\vec{P}|^2 = m^2c^2$$

This relation,

$$E^2 = p^2c^2 + m^2c^4,$$

is the cornerstone of relativistic particle physics and holds for both massive and massless particles.

1.11 Relativistic Form of Newton's Laws

In special relativity, Newton's second law is written using 4-vectors as:

$$\frac{dP^\mu}{d\tau} = F^\mu$$

where $P^\mu = mU^\mu$ is the 4-momentum, τ is proper time, and F^μ is the 4-force.

The spatial components of F^μ relate to the classical 3-force $\vec{f}$ via:

$$F^\mu = \left(\gamma\vec{f}\cdot\vec{v}/c,\ \gamma\vec{f}\right)$$

This ensures the spatial components match the Newtonian form in frame S:

$$\frac{d\vec{p}}{dt} = \vec{f}$$

and the temporal component becomes:

$$F^0 = \frac{1}{c}\frac{dE}{dt}$$

Using $P^\mu P_\mu = m^2c^2$, we can derive:

$$\frac{d}{d\tau}(P^\mu P_\mu) = 2P^\mu \frac{dP_\mu}{d\tau} = 2\gamma m \left(\frac{dE}{dt} - \vec{u} \cdot \vec{f} \right) = 0$$

which gives the energy-work relation.

1.12 Relativistic Electromagnetic Force

In relativistic electrodynamics, the equation of motion for a particle of charge q is:

$$\frac{dP^\mu}{d\tau} = \frac{q}{c} G^\mu{}_\nu U^\nu$$

where $G^{\mu\nu}$ is the electromagnetic field strength tensor:

$$G^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}$$

The spatial part of this equation recovers the Lorentz force law:

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

and the temporal part gives the power delivered:

$$\frac{dE}{dt} = q\vec{E} \cdot \vec{v}$$

1.13 Acceleration and the Rindler Horizon

In special relativity, describing accelerated motion is subtler than in Newtonian mechanics, since uniform acceleration in one frame does not generally translate the same way in another. The most natural concept is *proper acceleration* — the acceleration measured by an accelerometer moving with the object. It is invariant under Lorentz transformations and physically meaningful to the accelerated observer.

Hyperbolic Trajectories and Proper Acceleration

A particle experiencing constant proper acceleration a in flat spacetime traces a hyperbolic worldline in Minkowski space. Assuming motion in the x-direction, the trajectory as a function of proper time τ is:

$$x(\tau) = \frac{c^2}{a} \cosh\left(\frac{a\tau}{c}\right), \quad t(\tau) = \frac{c^2}{a} \sinh\left(\frac{a\tau}{c}\right)$$

This satisfies the invariant:

$$x^2 - c^2t^2 = \left(\frac{c^2}{a}\right)^2$$

indicating motion along a hyperbola. The proper acceleration remains constant in the co-moving instantaneous rest frame, even though coordinate acceleration decreases over time.

Rindler Coordinates and Observers

To describe the reference frame of an observer undergoing constant acceleration, we introduce Rindler coordinates (η, ξ), related to Minkowski coordinates (t, x) via:

$$x = \xi \cosh\left(\frac{a\eta}{c}\right), \quad t = \xi \sinh\left(\frac{a\eta}{c}\right)$$

Here, ξ is a spatial coordinate labeling the observer's position in the accelerated frame, and η plays the role of time for Rindler observers. The Minkowski metric transforms into:

$$ds^2 = -a^2\xi^2 d\eta^2 + d\xi^2 + dy^2 + dz^2$$

This metric shows that Rindler observers are in a non-inertial frame and experience a horizon at $\xi = 0$, where proper acceleration diverges.

Rindler Horizon and Causality

The Rindler horizon is a boundary in spacetime beyond which events cannot influence an accelerating observer. It arises naturally from the coordinate transformation: the accelerated observer's worldline asymptotically approaches the line $x = ct$, but never crosses it.

This leads to a causal structure similar to a black hole horizon. From the perspective of the accelerated observer, signals from beyond $x = ct$ (for right-moving observers) are forever unreachable.

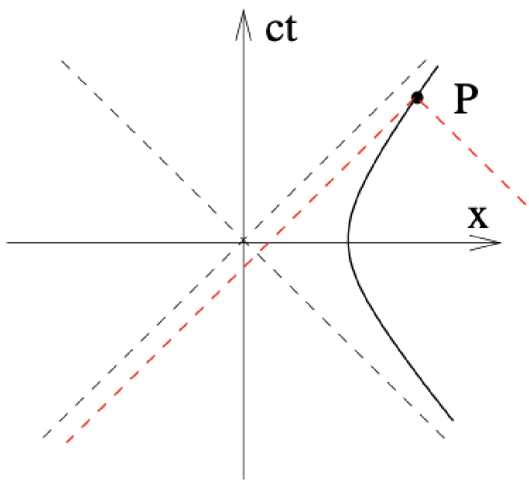

Figure 1: Spacetime diagram showing hyperbolic worldlines and the Rindler horizon.

Figure 2: Spacetime diagram showing hyperbolic worldlines and the Rindler horizon.

The Rindler horizon is not a physical barrier, but a limit on information accessibility due to the observer's accelerated frame. This concept is foundational in discussions of relativistic thermodynamics and the Unruh effect.

1.14 Raising and Lowering Indices

In special relativity, we often work with 4-vectors $X^\mu = (ct, x, y, z)$ and other tensors. Unlike in Euclidean space, the **Minkowski metric** $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ introduces a crucial distinction between **upper indices (contravariant)** and **lower indices (covariant)**.

Why the Index Position Matters

A 4-vector written with an upper index, X^μ, transforms differently than one with a lower index, X_μ. They are related by the Minkowski metric:

$$X_\mu = \eta_{\mu\nu} X^\nu$$

Explicitly, this means:

$$X^\mu = (ct, x, y, z) \quad \Rightarrow \quad X_\mu = (ct, -x, -y, -z)$$

Using this, the Minkowski inner product between two 4-vectors X^μ and X^ν can be compactly written using Einstein summation:

$$X^\mu X_\mu = \eta_{\mu\nu} X^\mu X^\nu = c^2t^2 - x^2 - y^2 - z^2$$

This expression is Lorentz invariant and reflects the correct spacetime geometry.

Important: It is incorrect and misleading to write $X^\mu X^\mu = c^2t^2 + x^2 + y^2 + z^2$ in the relativistic context. Such expressions are not invariant and break the formalism of special relativity.

The Role of the Minkowski Metric

The Minkowski metric $\eta_{\mu\nu}$ serves to raise and lower indices:

$$X^\mu = \eta^{\mu\nu} X_\nu, \quad X_\mu = \eta_{\mu\nu} X^\nu$$

Since $\eta_{\mu\nu} = \eta^{\mu\nu}$, raising and lowering operations use the same matrix.

Importantly, contractions (i.e., sums over repeated indices) must always involve one upper and one lower index:

$$A^\mu B_\mu = \text{valid and Lorentz invariant}, \quad A^\mu B^\mu = \text{invalid}$$

Example: The Electromagnetic Tensor

Consider the antisymmetric electromagnetic field strength tensor $G_{\mu\nu}$, defined in matrix form:

$$G_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

This object naturally lives with indices down. If needed, one can raise an index using the metric:

$$G^\mu{}_\nu = \eta^{\mu\rho} G_{\rho\nu}$$

The antisymmetry $G_{\mu\nu} = -G_{\nu\mu}$ is preserved under index raising.

Conceptual Significance

This distinction between upper and lower indices is more than notational:

- X^μ is a **contravariant vector** — it transforms like a coordinate displacement.
- X_μ is a **covariant vector** — it transforms like a gradient or 1-form.

In deeper mathematics (e.g., general relativity), these live in different vector spaces: the tangent space and its dual. The Minkowski metric provides a natural isomorphism between them.

By strictly following the index conventions and summing only over mixed pairs, we ensure that all expressions remain Lorentz invariant — a core requirement of special relativity.

2 General Relativity

General Relativity (GR), proposed by Albert Einstein in 1915, is a revolutionary theory that redefines gravity not as a force, but as a manifestation of the curvature of spacetime itself. It generalizes the principles of Special Relativity to include accelerated motion and incorporates gravity as a geometric property of spacetime.

2.1 Why Newton's Law of Gravity Fails

Newton's universal law of gravitation,

$$F = \frac{GMm}{r^2},$$

treats gravity as an instantaneous force acting at a distance. While successful in many practical contexts, Newton's theory breaks down both conceptually and experimentally under extreme conditions or high precision.

Conceptual Failures

- **Incompatibility with Special Relativity:** Newtonian gravity assumes instantaneous interaction, violating the principle that no information can travel faster than light.
- **Absolute Space and Time:** Newton's theory requires a fixed background of absolute space and time. Special relativity replaced this with a unified, observer-dependent spacetime, making Newton's formulation outdated.
- **No Role for Energy and Pressure:** In Newtonian gravity, only mass generates the gravitational field. But according to relativity, energy, momentum, and pressure (components of the stress-energy tensor) also gravitate.

Empirical Failures

- **Perihelion Precession of Mercury:** The orbit of Mercury deviates from Newtonian predictions. General relativity explains the extra precession precisely.
- **Deflection of Light by Gravity:** Newtonian gravity cannot account for the bending of light by massive objects, as light has no mass. GR predicts this as a result of spacetime curvature and was confirmed during the 1919 solar eclipse.
- **Gravitational Time Dilation and Redshift:** Effects involving time in gravitational fields (e.g. clocks ticking slower near Earth) are beyond Newtonian gravity and have been verified experimentally.

Towards a Field Theory of Gravity

In the relativistic view, gravity is not a force but a feature of spacetime geometry. Massive bodies curve spacetime, and objects follow geodesics — the straightest paths — within that curved geometry. This insight naturally leads us to describe gravity using a *field*: the metric tensor $g_{\mu\nu}$, which varies across spacetime.

The gravitational field is not mediated by forces in space but encoded in the structure of spacetime itself:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

The change in geometry caused by matter is governed by the Einstein field equations, which relate curvature to the distribution of energy and momentum:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

This framework unifies gravitation and geometry, laying the foundation for understanding black holes, gravitational waves, and the large-scale structure of the universe.

2.2 Electromagnetism vs Gravity and the Role of the Energy-Momentum Tensor

In classical field theory, both gravity and electromagnetism are long-range interactions, but their treatments and physical interpretations are fundamentally different — especially in the context of relativity.

Invariance of Charge vs Invariance of Mass

Electric charge is an intrinsic property of particles and is strictly invariant under Lorentz transformations. No matter how fast an observer moves, the measured electric charge of a particle remains the same. This is unlike energy or momentum, which vary with the frame.

Mass, or more precisely rest mass, is also invariant. But gravity — unlike electromagnetism — couples not just to mass, but to the **entire energy-momentum content** of matter. This includes:

- Energy (including mass-energy)
- Momentum

- Pressure
- Stress (momentum flow or momentum flux)

This is a core reason why gravity behaves differently from other forces in relativistic field theory.

Electromagnetism: A Gauge Field on Flat Spacetime

The electromagnetic field A^μ lives in flat Minkowski spacetime, and the dynamics of the field are governed by Maxwell's equations. The field equations are:

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$

where:

- $F^{\mu\nu}$ is the electromagnetic field tensor,
- $J^\mu = (\rho c, \mathbf{j})$ is the 4-current (charge and current),
- ∂_ν are partial derivatives in flat spacetime.

Electric charge appears as a source term, and is invariant. The spacetime remains fixed — electromagnetic fields evolve *on* spacetime, not *with* it.

Gravity: Geometry as Dynamics

In contrast, general relativity makes spacetime itself dynamical. The "field" is the metric $g_{\mu\nu}$, and it evolves according to Einstein's equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The object $G_{\mu\nu}$, called the **Einstein tensor**, encodes how spacetime is curved. It is defined by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

where:

- $R_{\mu\nu}$ is the Ricci tensor, derived from spacetime curvature.
- R is the Ricci scalar, a trace of $R_{\mu\nu}$.
- $g_{\mu\nu}$ is the metric tensor.

The Einstein tensor is symmetric and satisfies $\nabla^\mu G_{\mu\nu} = 0$, ensuring local conservation of energy and momentum.

The Stress-Energy Tensor $T_{\mu\nu}$

The stress-energy tensor is a symmetric rank-2 tensor:

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & -cj_x & -cj_y & -cj_z \\ -cj_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ -cj_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ -cj_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Where:

- ρ: energy density
- j_i: momentum density or energy flux
- σ_{ij}: stress components or **momentum flux**, representing the rate at which the i-th component of momentum flows across a surface normal to the j-th direction.

This tensor is the source term in the Einstein field equations — it tells spacetime how to curve.

Key Differences Between Electromagnetism and Gravity

- Electromagnetism acts in a fixed spacetime; gravity changes the structure of spacetime itself.
- The electromagnetic field couples to charge (a scalar invariant); gravity couples to $T_{\mu\nu}$, a tensor involving all forms of energy and momentum.
- Electromagnetic waves propagate over spacetime; gravitational waves are ripples *in* spacetime.

2.3 Geodesics in Spacetime

In general relativity, particles not subject to any non-gravitational forces move along geodesics — the straightest possible paths in curved spacetime. Just as in classical mechanics, this path can be obtained by applying the **principle of least action**.

Principle of Least Action

The path that a free particle follows between two spacetime events is the one that extremizes the action:

$$S = \int \mathcal{L}\, d\lambda$$

where λ is an arbitrary parameter along the path, and $\mathcal{L}$ is the Lagrangian.

For a free particle in curved spacetime, the natural choice of action is proportional to the proper time:

$$S = -m \int ds = -m \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

We can simplify this by choosing the Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$

which gives the same equations of motion due to reparametrization invariance (since the square root form and this form extremize the same path).

Geodesic Equation

Using the Euler–Lagrange equations:

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0$$

with $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$, we compute:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = g_{\alpha\nu}\dot{x}^\nu, \qquad \frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

Plugging into the Euler–Lagrange equation gives:

$$\frac{d}{d\lambda}(g_{\alpha\nu}\dot{x}^\nu) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

After manipulating and using the definition of the Christoffel symbols:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right)$$

we obtain the **geodesic equation**:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$$

This equation describes the path of a free-falling particle through curved spacetime.

Non-Relativistic Limit

In the non-relativistic (Newtonian) limit:

- Velocities are small: $|\vec{v}| \ll c$
- The gravitational field is weak: spacetime is nearly flat.
- The dominant metric perturbation is $g_{00} \approx 1 + 2\phi/c^2$, where ϕ is the Newtonian gravitational potential.

In this limit, the geodesic equation reduces to Newton's second law:

$$\frac{d^2x^i}{dt^2} = -\partial^i\phi$$

This shows that Newtonian gravity emerges as the low-speed, weak-field approximation of general relativity.

A Particle in Minkowski Spacetime

Consider a particle moving in Minkowski spacetime $R^{1,3}$, with Cartesian coordinates $x^\mu = (ct, x, y, z)$, and metric:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$$

The invariant spacetime interval between two infinitesimally separated points is:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$$

Depending on the sign of ds^2, events are:

- **Timelike** if $ds^2 < 0$
- **Spacelike** if $ds^2 > 0$
- **Lightlike (null)** if $ds^2 = 0$

To describe the trajectory of a particle, we use a parameter σ, which increases monotonically along the worldline, with endpoints σ_1 and σ_2. The action for a free massive particle (with rest mass m) is:

$$S = -mc \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}$$

This action has dimensions of energy $\times$ time, as required, and geometrically represents the proper time along the timelike worldline.

Symmetries of the Action

- **Lorentz Invariance:** The action is invariant under Lorentz transformations,

  $$x^\mu \to \Lambda^\mu{}_\rho x^\rho \quad \text{with} \quad \Lambda^\mu{}_\sigma \eta_{\mu\nu} \Lambda^\nu{}_\rho = \eta_{\sigma\rho}$$

  Solutions to the equations of motion are mapped to new solutions by Lorentz transformations.

- **Reparameterisation Invariance:** The parameter σ is arbitrary. If we change variables to a new monotonic parameter $\tilde{\sigma}(\sigma)$, the form of the action remains unchanged:

  $$S = -mc \int_{\tilde{\sigma}_1}^{\tilde{\sigma}_2} d\tilde{\sigma} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tilde{\sigma}} \frac{dx^\nu}{d\tilde{\sigma}}}$$

  This is not a symmetry that generates new solutions, but a **redundancy** — similar to gauge invariance — reflecting that physics does not depend on how we parameterise the worldline.

Degrees of Freedom and the Role of Parameterisation At first glance, the relativistic action appears to involve four dynamical degrees of freedom $x^\mu(\sigma)$, in contrast to the three spatial degrees of freedom $x^i(t)$ in the non-relativistic case. This raises the question: are we introducing more physical degrees of freedom by adopting a relativistic description?

The answer lies in **reparameterisation invariance**. Because the parameter σ is arbitrary and carries no physical meaning, not all four functions $x^\mu(\sigma)$ are independent. The actual trajectory is defined by the relation between the coordinates x^μ, not by their individual parameterisations. Thus, one degree of freedom is redundant, and we are left with three physical degrees of freedom — just as in the non-relativistic case.

As a concrete example, we can choose a specific parameterisation $\sigma = t$, where t is the time measured by some inertial observer. The action becomes:

$$S = -mc \int_{t_1}^{t_2} dt \sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

where $\dot{x} = \frac{dx}{dt}$. This is the action for a relativistic particle in a particular frame, which makes explicit the appearance of the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}$$

While this form reveals clearly that the particle has three physical degrees of freedom $x(t)$, it obscures the Lorentz invariance of the original covariant action. That symmetry becomes manifest only when all spacetime coordinates are treated on equal footing.

Rediscovering the Forces of Nature

So far, we've only considered the action for a free relativistic particle. To include forces, we must add terms to the action — but in a way that preserves **reparameterisation invariance**.

In non-relativistic mechanics, forces are introduced via a potential:

$$S_{\text{non-rel}} = \int dt \left(\frac{1}{2} m\dot{x}^2 - V(x) \right)$$

However, in the relativistic case, naively adding a potential term $\int d\sigma \, V(x)$ breaks reparameterisation invariance. To maintain this symmetry, we must construct interaction terms that cancel any Jacobian from reparametrisation.

Introducing a Vector Field $A_\mu(x)$ To retain Lorentz invariance and reparameterisation invariance, we consider adding a term linear in the velocity:

$$S = -mc \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \;-\; q \int_{\sigma_1}^{\sigma_2} d\sigma \, A_\mu(x) \frac{dx^\mu}{d\sigma}$$

Here:

- $A_\mu(x)$ is a spacetime-dependent vector field,
- q is a coupling constant, later interpreted as electric charge.

This form ensures both reparameterisation and Lorentz invariance.

Recovering the Electromagnetic Interaction Choosing $\sigma = t$ and writing $A^\mu(x) = \left(\frac{\phi(x)}{c}, \mathbf{A}(x) \right)$, the action becomes:

$$S = \int dt \left(-mc^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} - q\phi(x) + q\mathbf{A}(x) \cdot \dot{\mathbf{x}} \right)$$

This is precisely the action for a charged particle in an electromagnetic field. The term involving $\phi(x)$ gives rise to the electric potential energy, while $\mathbf{A}(x) \cdot \dot{\mathbf{x}}$ introduces magnetic effects.

Thus, by demanding the correct symmetries of relativistic mechanics, we naturally rediscover the **Lorentz force law** of electromagnetism.

The Equivalence Principle and Gravitational Action

Let us consider the non-relativistic limit of a relativistic particle moving under a potential $V(x) = m\Phi(x)$, where $\Phi(x)$ is the Newtonian gravitational potential. The action takes the approximate form:

$$S \approx \int dt \left(\frac{1}{2} m\dot{x}^2 - m\Phi(x) \right)$$

We immediately notice a key feature: the mass m appears in both the kinetic and potential terms. This implies that the strength of the gravitational force is proportional to the inertial mass.

In Newtonian mechanics, the two masses need not be equal. One may distinguish between:

- **Inertial mass** m_I: coefficient in kinetic energy.
- **Gravitational mass** m_G: coupling to the gravitational field.

But experiments show that:

$$m_I = m_G$$

to extraordinary precision (10^{-13}). This empirical fact is known as the **equivalence principle**.

A Geometric Interpretation Our relativistic action provides a natural explanation: since the mass multiplies the entire action uniformly, no distinction arises between m_I and m_G. Gravity does not act like a force in the traditional sense — instead, it modifies the background geometry in which particles move.

To capture this in the action, we promote the flat metric component $\eta_{00} \approx -1$ to a weak-field expansion:

$$\eta_{00} \to g_{00}(x) \approx -\left(1 + \frac{2\Phi(x)}{c^2}\right)$$

But modifying only η_{00} would break Lorentz symmetry. To preserve covariance, we promote all components of $\eta_{\mu\nu}$ to spacetime-dependent functions: a full metric $g_{\mu\nu}(x)$. This leads us to the reparameterisation invariant gravitational action:

$$S = -mc \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}$$

This describes a particle moving in curved spacetime — not under a force, but following geodesics dictated by the geometry. Thus, the equivalence principle naturally guides us to general relativity: gravity as curvature.

The Equivalence Principle

Equivalence Principle (Weak Form): Locally, the effects of a gravitational field are indistinguishable from those of constant acceleration. That is, an observer in free fall cannot detect the presence of gravity by any local experiment.

This implies that gravity and inertia are fundamentally linked — and that freely falling frames are locally inertial. This forms the foundational idea of general relativity.

Example: A person in a windowless elevator cannot distinguish between:

- standing still in Earth's gravity,
- or accelerating upward in deep space at $a = g$.

Rapidity: In special relativity, **rapidity** θ replaces velocity as a more natural additive parameter for boosts.

For motion along one spatial axis:

$$v = c\tanh\theta, \quad \gamma = \cosh\theta, \quad \gamma v/c = \sinh\theta$$

Thus, a Lorentz boost becomes:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Rapidity is additive under successive boosts, unlike velocity. This makes it a more natural parameter when composing Lorentz transformations.

Uniform Acceleration and the Equivalence Principle

In special relativity, it is often more convenient to parametrize motion using **rapidity** φ, which is related to velocity via

$$v = c\tanh\varphi$$

Rapidity has the nice property of being additive under successive Lorentz boosts:

$$\varphi = \varphi_1 + \varphi_2$$

Uniform Acceleration For an observer undergoing constant proper acceleration a, the rapidity increases linearly with proper time τ:

$$\varphi(\tau) = \frac{a\tau}{c} \quad \Rightarrow \quad v(\tau) = c\tanh\left(\frac{a\tau}{c}\right)$$

This yields a relation between proper time τ and coordinate time t in the inertial frame:

$$\frac{dt}{d\tau} = \gamma(\tau) = \cosh\left(\frac{a\tau}{c}\right) \quad \Rightarrow \quad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

$$x(\tau) = \frac{c^2}{a}\cosh\left(\frac{a\tau}{c}\right) - \frac{c^2}{a}$$

This describes a **hyperbolic trajectory** in Minkowski spacetime:

$$\left(x + \frac{c^2}{a}\right)^2 - c^2t^2 = \left(\frac{c^2}{a}\right)^2$$

Accelerated Coordinates and Rindler Spacetime: From the perspective of the accelerating observer, it is natural to define coordinates (τ, ρ) such that the observer sits at $\rho = 0$. The transformation from inertial coordinates (ct, x) is:

$$ct = \left(\rho + \frac{c^2}{a}\right)\sinh\left(\frac{a\tau}{c}\right), \quad x = \left(\rho + \frac{c^2}{a}\right)\cosh\left(\frac{a\tau}{c}\right) - \frac{c^2}{a}$$

These coordinates cover only the right wedge of Minkowski space — the region accessible to the accelerating observer. This limited access is associated with the concept of an **event horizon**.

Rindler Metric: Substituting into the flat Minkowski line element:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

yields:

$$ds^2 = -\left(1 + \frac{a\rho}{c^2}\right)^2 c^2 d\tau^2 + d\rho^2 + dy^2 + dz^2$$

This is the **Rindler metric**, describing flat spacetime from the viewpoint of a uniformly accelerating observer. The spatial sections remain flat, but the time component now depends on ρ.

Gravitational Interpretation: The metric in the coordinates (τ, ρ), known as **Kottler–Møller coordinates**, is a rewriting of part of flat Minkowski spacetime from the viewpoint of a uniformly accelerating observer. It reads:

$$ds^2 = -\left(1 + \frac{a\rho}{c^2}\right)^2 c^2 d\tau^2 + d\rho^2 + dy^2 + dz^2$$

This is closely related to the **Rindler metric**, which we will study later in the context of black hole horizons.

The spatial part of the metric remains flat, but the temporal component gains a spatial dependence:

$$g_{00} = -\left(1 + \frac{a\rho}{c^2}\right)^2 = -\left(1 + \frac{2a\rho}{c^2} + \frac{a^2\rho^2}{c^4}\right)$$

This expansion matches the expected form from Newtonian gravity:

$$g_{00} \approx -\left(1 + \frac{2\Phi(\rho)}{c^2}\right) \quad \Rightarrow \quad \Phi(\rho) = a\rho$$

Interpretation: The accelerating observer perceives a linearly increasing gravitational potential $\Phi(\rho) = a\rho$, as if immersed in a constant gravitational field. This is a direct realization of the **equivalence principle**: acceleration and gravity are locally indistinguishable.

The observer's experience in the accelerating frame (with metric above) is equivalent to being in a gravitational field with uniform strength a, even though they are in flat spacetime. This geometric reformulation of inertial effects lies at the heart of general relativity.

Einstein Equivalence Principle and Tidal Forces: The **Einstein Equivalence Principle** states that in any sufficiently small region of spacetime, there always exists a freely falling (locally inertial) frame in which the effects of gravity vanish. Mathematically, this means the metric takes the form

$$g_{\mu\nu} \approx \eta_{\mu\nu} \quad \text{and} \quad \Gamma^\lambda_{\mu\nu} = 0$$

at a single point. These coordinates are those of a freely falling observer, where locally, spacetime appears flat.

However, this equivalence is only *local*. Over an extended region, the metric can no longer be transformed into Minkowski form everywhere, and curvature effects emerge through non-zero second derivatives of the metric.

A clear manifestation of this is seen through **tidal forces**. Suppose you are in a weightless elevator with two small test masses placed near each other. If you are in true inertial motion (e.g., drifting in flat space), the balls remain stationary relative to each other. But in a non-uniform gravitational field (e.g., falling toward Earth), their geodesics begin to *diverge or converge*.

This deviation signals spacetime curvature and is captured by the **geodesic deviation equation**. Thus, while gravity can be eliminated locally by choosing an appropriate frame, curvature—and hence gravity—can still be detected over finite regions.

Gravitational Time Dilation:

Even before solving Einstein's equations, we can already understand how gravity affects time. In a weak gravitational field $\Phi(x)$, the temporal component of the metric becomes:

$$g_{00}(x) = 1 + \frac{2\Phi(x)}{c^2}$$

For a spherically symmetric mass M, the Newtonian potential is:

$$\Phi(r) = -\frac{GM}{r}$$

Substituting into the metric gives:

$$d\tau^2 = g_{00}(r)\, dt^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2$$

An observer at radius r thus experiences a slower passage of time compared to someone far from the mass. If a distant observer measures time t, the local time experienced near the mass is:

$$T(r) = t\sqrt{1 - \frac{2GM}{rc^2}}$$

This phenomenon is known as **gravitational time dilation**.

The gravitational time dilation effect becomes dramatically more pronounced near a black hole. In Section 1.3, we will see that the closest stable circular orbit a planet can maintain around a non-rotating (Schwarzschild) black hole is at

$$r = \frac{3GM}{c^2}$$

At this radius, time passes significantly slower. A clock orbiting at this distance ticks at the rate:

$$T = t\sqrt{\frac{1}{3}} \approx 0.58t$$

compared to an asymptotic observer at infinity.

Why is this the innermost stable circular orbit (ISCO)? The answer lies in the nature of geodesics in curved spacetime. As one moves closer to the black hole, the effective potential for orbital motion becomes increasingly steep and unstable. Below $r = 3GM/c^2$, circular orbits are no longer stable — small perturbations in radius grow, causing the particle to spiral inward toward the event horizon. The ISCO marks the boundary between stable and unstable orbital motion.

In *Interstellar*, the planet is depicted as experiencing time so slow that one hour there corresponds to seven years on a distant spaceship. For this extreme time dilation to occur, the planet would need to be located much closer to the event horizon, near

$$r \approx \frac{2GM}{c^2}$$

However, this is already inside the ISCO, meaning no stable orbit is possible there for a Schwarzschild black hole. To achieve both stability and such extreme time dilation, one must consider a rapidly spinning (Kerr) black hole. In Kerr spacetime, frame dragging allows stable orbits to exist closer to the horizon, but even then, such orbits would be subject to intense tidal forces, radiation, and other hostile effects — making the setting in the movie highly idealized.

Thus, while the concept of gravitational time dilation in *Interstellar* is based on real physics, the magnitude and survivability of such a planet are scientifically implausible without invoking a very carefully tuned rotating black hole model.

Gravitational time dilation has also been experimentally confirmed using atomic clocks at different altitudes on Earth, in agreement with general relativity.

Gravitational Redshift:

Gravitational redshift is one of the key observable consequences of general relativity, arising due to the variation of the flow of time in a gravitational field. When light or any signal climbs out of a gravitational potential well, it loses energy, which manifests as a shift toward longer wavelengths (lower frequencies).

Consider a static observer at radius r in a weak gravitational field, where the spacetime metric is approximately:

$$g_{00}(r) \approx 1 + \frac{2\Phi(r)}{c^2}$$

Here, $\Phi(r)$ is the Newtonian potential, e.g., for a spherical mass M,

$$\Phi(r) = -\frac{GM}{r}$$

Suppose a photon is emitted from radius r_1 and received at a distant point r_2. The time dilation experienced by the emitting and receiving observers leads to a shift in the observed frequency:

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}}$$

In the weak-field limit ($\Phi/c^2 \ll 1$), this becomes:

$$\frac{\nu_2}{\nu_1} \approx 1 + \frac{\Phi(r_2) - \Phi(r_1)}{c^2}$$

$$\Rightarrow \frac{\Delta\nu}{\nu} = \frac{\nu_2 - \nu_1}{\nu_1} \approx \frac{\Phi(r_2) - \Phi(r_1)}{c^2}$$

If $r_2 > r_1$, i.e., the photon climbs out of the potential well, then $\Phi(r_2) > \Phi(r_1)$, so $\nu_2 < \nu_1$: a redshift. Conversely, light falling into a potential well will be blueshifted.

Experimental Verification: Gravitational redshift has been confirmed experimentally through observations such as:

- The Pound-Rebka experiment (1959), which measured redshift over a height of 22.5 meters using gamma rays.
- Observations of spectral lines from white dwarfs and neutron stars.
- GPS satellites, which must account for both gravitational and special relativistic time dilations to maintain accuracy.

Gravitational redshift directly ties the geometry of spacetime to observational phenomena, reinforcing the idea that gravity is not a force in the Newtonian sense, but a manifestation of curved spacetime.

Geodesics in Spacetime:

So far, we have focussed entirely on the actions describing particles, and have not yet written down an equation of motion, let alone solved one. Now it's time to address this.

We consider the relativistic action for a particle moving in curved spacetime:

$$S = -mc \int_{\sigma_1}^{\sigma_2} d\sigma\, \mathcal{L}, \quad \text{with} \quad \mathcal{L} = \sqrt{-g_{\mu\nu}(x)\, \dot{x}^\mu\, \dot{x}^\nu}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\sigma}$ is the derivative of the particle's worldline with respect to an arbitrary parameter σ. This form is similar to the non-relativistic geodesic action, but the square root introduces complications when deriving the equations of motion.

Equations of Motion: We compute the Euler-Lagrange equations:

$$\frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\rho}\right) - \frac{\partial \mathcal{L}}{\partial x^\rho} = 0$$

Working through the algebra, we obtain:

$$\frac{d}{d\sigma}\left(\frac{g_{\rho\nu}\dot{x}^\nu}{\mathcal{L}}\right) - \frac{1}{2\mathcal{L}}\partial_\rho g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

This yields a complicated form due to the square-root factor $\mathcal{L}$. However, this can be greatly simplified if we choose to reparameterise the worldline using the *proper time* τ, which is defined by:

$$cd\tau = \mathcal{L}(\sigma)\, d\sigma$$

This ensures that $\mathcal{L} = c$, and thus $\frac{d\mathcal{L}}{d\tau} = 0$. Such parameters related linearly to proper time ($\tilde{\tau} = a\tau + b$) are called *affine parameters.*

The Geodesic Equation: Using an affine parameter, the Euler-Lagrange equations simplify to the relativistic geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right)$$

This is the most important equation describing how particles move under gravity in general relativity: they follow geodesics, the "straightest possible paths" in curved spacetime.

A Useful Trick: Rather than dealing with square roots, we can use a simplified form of the action (when working purely classically):

$$S_{\text{useful}} = \int d\tau\, g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

This will produce the same geodesic equation (up to affine reparameterisation), provided we impose the constraint:

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2$$

which ensures the worldline is timelike.

Massless Particles: For massless particles, such as photons, the path followed is a *null geodesic*, and the constraint becomes:

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

The fact that the geodesic equation is independent of mass reflects the equivalence principle: all objects, regardless of mass, fall the same way in a gravitational field.

This derivation shows how geodesic motion emerges naturally from the action principle in general relativity, and how proper time plays a central role in simplifying the formalism.

A Useful Trick:

We started in Section 1.1.1 with the non-relativistic action

$$S = \int dt\, \frac{m_{ij}}{2} g_{ij}(x)\dot{x}^i\dot{x}^j$$

and found that it gives rise to the geodesic equation (1.7).

However, to describe relativistic physics in spacetime, we need to incorporate *reparameterisation invariance* into our formalism. This leads us to the action

$$S = -mc\int d\sigma\, \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}$$

Nonetheless, when we restrict to a particular parameterisation — the proper time τ — we find exactly the same geodesic equation (1.30) that we encountered in the non-relativistic case.

This suggests a shortcut. If our goal is merely to derive the geodesic equation for a given metric, we can work with the simplified action:

$$S_{\text{useful}} = \int d\tau \, g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \tag{1.32}$$

This will give the desired equations of motion, provided it is supplemented with the constraint:

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2 \tag{1.33}$$

This constraint enforces that the geodesic is timelike, with τ representing proper time. It ensures that the particle moves forward in time. Notably, neither the action (**??**) nor the constraint (**??**) depend on the mass m of the particle. This is a reflection of the *equivalence principle*, which tells us that all particles, regardless of mass, follow the same geodesic.

Moreover, we can also use (**??**) to compute the motion of massless particles such as light. These follow *null geodesics*, which means we replace the constraint with:

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{1.34}$$

While the action S_{useful} is, as the name suggests, useful, it must be wielded with caution. As written, it does not have the correct physical dimensions for an action. Furthermore, if one tries to use it in the context of quantum mechanics or statistical mechanics, improper handling of the constraint could lead to incorrect results.

A First Look at the Schwarzschild Metric

Physics was born from our attempts to understand the motion of the planets. The problem was largely solved by Newton, who was able to derive Kepler's laws of planetary motion from the gravitational force law. This was described in some detail in our first lecture course on Dynamics and Relativity.

Newton's laws are not the end of the story. There are relativistic corrections to the orbits of the planets that can be understood by computing the geodesics in the background of a star.

To do this, we first need to understand the metric created by a star. This will be derived in Section 6. For now, we simply state the result: a star of mass M gives rise to a curved spacetime described by the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{1}$$

The coordinates θ and ϕ are the usual spherical polar coordinates, with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

We can already perform a few sanity checks. First, note that far from the star, as $r \to \infty$, the metric reduces to the flat Minkowski metric, as it should. Secondly, the g_{00} component is

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right), \quad \text{with} \quad \Phi(r) = -\frac{GM}{r} \tag{2}$$

which matches our earlier expectation for a weak gravitational potential.

However, the Schwarzschild metric also has some curious features. In particular, the g_{rr} component diverges at

$$r = R_s = \frac{2GM}{c^2} \tag{3}$$

This radius R_s is known as the Schwarzschild radius. It marks the location of an event horizon if the star is sufficiently compact — in other words, if it's a black hole. This phenomenon will be studied more deeply in Section 6.

That said, the Schwarzschild metric describes spacetime outside any spherically symmetric mass distribution, whether it's a black hole or a regular star. For a regular star, the metric applies only outside the stellar surface $r > R_{\text{star}}$, where $R_{\text{star}} \gg R_s$.

In what follows, we will treat planets as test particles moving along geodesics in the Schwarzschild geometry and study the relativistic corrections to planetary orbits and other physical effects in this spacetime.

Geodesics in the Schwarzschild Background

Our first task is to derive the equations for a geodesic in the Schwarzschild background. To do this, we use the quick and easy method of looking at the action (1.32) for a particle moving in Schwarzschild spacetime,

$$S_{\text{useful}} = \int d\tau\, L = \int d\tau\, g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu \tag{4}$$

$$= \int d\tau \left[-A(r)c^2\dot{t}^2 + A^{-1}(r)\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)\right] \tag{5}$$

where $A(r) = 1 - \frac{R_s}{r}$ and $\dot{x}^\mu = dx^\mu/d\tau$.

Just like in the Newtonian Kepler problem, we can use conservation of angular momentum to restrict the motion to a plane. Consider the equation of motion for θ:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \Rightarrow \quad \frac{d}{d\tau}(r^2\dot{\theta}) = r^2 \sin\theta\cos\theta\, \dot{\phi}^2 \tag{6}$$

This tells us that if we choose initial conditions with $\theta = \pi/2$ and $\dot{\theta} = 0$, then the motion will remain confined to the equatorial plane. We adopt this simplification.

Conserved Quantities: Next, we exploit the symmetries of the Schwarzschild metric to identify conserved quantities. Since the Lagrangian is independent of both t and ϕ, we apply the Euler-Lagrange equation:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \quad \Rightarrow \quad \text{conserved quantity} \tag{7}$$

The conserved quantity associated with ϕ (angular momentum per unit mass) is:

$$2\ell = \frac{\partial L}{\partial \dot{\phi}} = 2r^2\dot{\phi} \quad \Rightarrow \quad \ell = r^2\dot{\phi} \tag{8}$$

The conserved quantity associated with t (energy per unit mass) is:

$$-2E = \frac{\partial L}{\partial \dot{t}} = -2A(r)c^2\dot{t} \quad \Rightarrow \quad E = A(r)c^2\dot{t} \tag{9}$$

As $r \to \infty$, where $A(r) \to 1$, we return to flat space. There, we know $\dot{t} = \gamma$, and $E \to \gamma c^2$, consistent with the energy per unit rest mass in special relativity.

Geodesic Constraint: We now impose the proper time constraint from (1.33). Setting $\theta = \pi/2$ and $\dot{\theta} = 0$, we get:

$$-A(r)c^2\dot{t}^2 + A^{-1}(r)\dot{r}^2 + r^2\dot{\phi}^2 = -c^2 \tag{10}$$

Substituting the conserved quantities for $\dot{t}$ and $\dot{\phi}$, we find:

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}\frac{E^2}{c^2} \tag{11}$$

with the effective potential

$$V_{\text{eff}}(r) = \frac{1}{2}\left(c^2 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{R_s}{r}\right) \tag{12}$$

This is the key equation governing the radial motion of the particle. Once the radial solution is known, we can reconstruct the full orbit using the expression for angular momentum.

2.4 Planetary Orbits in General Relativity

We now repeat this analysis for the full relativistic motion of a massive particle moving along a geodesic in the Schwarzschild metric. We have seen that the effective potential takes the form:

$$V_{\text{eff}}(r) = \frac{c^2}{2} - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3c^2} \tag{13}$$

The relativistic correction scales as $-1/r^3$ and modifies the Newtonian story at short distances, ensuring that the potential $V_{\text{eff}}(r) \to -\infty$ as $r \to 0$. Notably, the potential vanishes at the Schwarzschild radius $r = R_s = 2GM/c^2$, with $V_{\text{eff}}(R_s) = 0$.

The shape of the potential depends on the angular momentum. To analyze this, we compute the critical points:

$$V'_{\text{eff}}(r) = \frac{GM}{r^2} - \frac{l^2}{r^3} + \frac{3GMl^2}{r^4c^2} = 0 \tag{14}$$

This equation can be rearranged as:

$$GMr^2 - l^2r + \frac{3GMl^2}{c^2} = 0 \tag{15}$$

This quadratic equation in r has two solutions when the discriminant is positive, i.e., when

$$l^2 > \frac{12G^2M^2}{c^2} \tag{16}$$

In this case, the effective potential has a local maximum and a local minimum. Let us denote the solutions to the quadratic as r_+ and r_- with $r_+ > r_-$. The outer root r_+ corresponds to a stable circular orbit; the inner root r_- corresponds to an unstable circular orbit.

There also exist non-circular orbits that oscillate around the minimum of the potential. Unlike Newtonian orbits, these are not necessarily closed or elliptical. A notable feature is that the angular momentum barrier is now finite: even a particle with large angular momentum can fall into the center if it has sufficient energy.

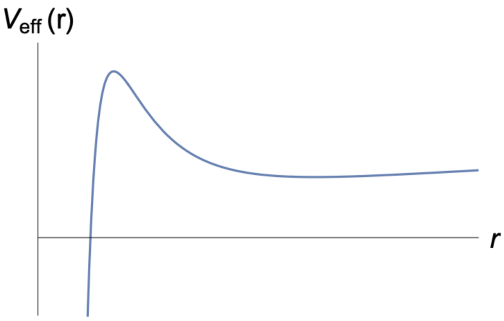

Figure 4: The effective potential for a massive particle when $l^2c^2 > 12G^2M^2$.

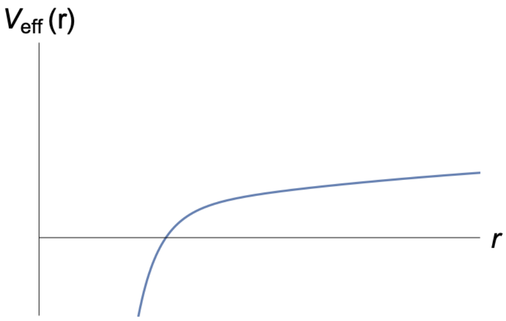

Figure 5: ...and when $l^2c^2 < 12G^2M^2$.

If $l^2 < \frac{12G^2M^2}{c^2}$, then $V_{\text{eff}}(r)$ has no turning points and the potential is monotonically decreasing. No stable orbits exist in this case; all particles eventually fall inward.

The borderline case $l^2 = \frac{12G^2M^2}{c^2}$ corresponds to a saddle point in the potential, located at:

$$r_{\text{ISCO}} = \frac{6GM}{c^2} \tag{17}$$

This is known as the Innermost Stable Circular Orbit (ISCO). For $r < r_{\text{ISCO}}$, no circular orbits exist, though non-circular trajectories may still dip into this region.

The ISCO plays a significant role in black hole astrophysics, especially as it marks the inner edge of an accretion disk. In observations like the Event Horizon Telescope image, the ring of light roughly corresponds to the photon sphere rather than the ISCO. Nonetheless, the ISCO still defines the last possible stable orbit for matter.

Finally, we can ask: how close can a non-circular orbit approach the black hole? In the limit $l \to \infty$, the maximum of the effective potential approaches:

$$r_- \to \frac{3GM}{c^2} \tag{18}$$

This is the closest distance that any timelike geodesic can reach and still escape, making it an important threshold in relativistic dynamics.

2.5 Perihelion Precession

To analyze the relativistic correction to planetary orbits, we introduce the inverse radial coordinate

$$u = \frac{1}{r}$$

and express the equations of motion in terms of $u(\phi)$. Starting from the conserved energy and angular momentum, and applying general relativistic corrections, the equation of motion becomes

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2GM}{l^2}u - \frac{2GM}{c^2}u^3 = \frac{E^2}{l^2c^2} - \frac{1}{l^2}$$

This is considerably more complex than the Newtonian orbital equation. To simplify, we differentiate both sides with respect to ϕ, assuming $\frac{du}{d\phi} \neq 0$ (i.e., excluding circular orbits), yielding:

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{l^2} + \frac{3GM}{c^2}u^2$$

This equation differs from the Newtonian result by the additional nonlinear term $\frac{3GM}{c^2}u^2$, which vanishes as $c \to \infty$.

To solve this, we employ perturbation theory by introducing the small parameter

$$\beta = \frac{3G^2M^2}{l^2c^2}$$

and seek a solution as a power series expansion in β:

$$u(\phi) = u_0(\phi) + \beta u_1(\phi) + \beta^2 u_2(\phi) + \cdots$$

At leading order ($\beta \to 0$), we retrieve the Newtonian orbit:

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{GM}{l^2} \quad \Rightarrow \quad u_0(\phi) = \frac{GM}{l^2}(1 + e\cos\phi)$$

Substituting u_0 into the right-hand side of the full equation, we obtain the inhomogeneous differential equation for u_1:

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{GM}{l^2}\,(1 + e\cos\phi)^2$$

Expanding the right-hand side:

$$(1 + e\cos\phi)^2 = 1 + 2e\cos\phi + e^2\cos^2\phi = 1 + 2e\cos\phi + \frac{e^2}{2} + \frac{e^2}{2}\cos(2\phi)$$

So the equation becomes:

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{GM}{l^2}\left(1 + \frac{e^2}{2} + 2e\cos\phi + \frac{e^2}{2}\cos(2\phi)\right)$$

This has the solution:

$$u_1(\phi) = \frac{GM}{l^2}\left(1 + \frac{e^2}{2} + e\phi\sin\phi - \frac{e^2}{6}\cos(2\phi)\right)$$

The key term here is $e\phi\sin\phi$, which is **not periodic** in ϕ and causes the orbit to not close after a full 2π revolution. This manifests as a slow **precession of the perihelion** (or aphelion) of the orbit.

The perihelion shift $\Delta\phi$ per revolution, to leading order in β, is given by:

$$\Delta\phi \approx 2\pi\beta = \frac{6\pi G^2M^2}{l^2c^2}$$

This is a classical test of general relativity, famously explaining the anomalous precession of Mercury's orbit.

Application to Mercury and Observable Shift

For planets orbiting the Sun, the perihelion shift depends only on the planet's angular momentum l and the mass of the Sun, denoted $M_\odot$. The latter is approximately

$$M_\odot \approx 2 \times 10^{30} \text{ kg} \quad \Rightarrow \quad \frac{GM_\odot}{c^2} \approx 1.5 \times 10^3 \text{ m}$$

For a planet in an almost circular orbit of radius r and orbital period T, the angular momentum is

$$l = \frac{2\pi r^2}{T}$$

Using Kepler's third law ($T \propto r^{3/2}$), it follows that $l \propto r^{1/2}$, and hence

$$\delta \propto \frac{1}{l^2} \propto \frac{1}{r}$$

This implies that **the perihelion shift is more significant for planets closer to the Sun**.

The innermost planet, **Mercury**, is ideal for observing this effect. Mercury's orbit has a notable eccentricity $e \approx 0.2$, with radial distance varying between

$$r_- \approx 4.6 \times 10^{10} \text{ m}, \quad r_+ \approx 7 \times 10^{10} \text{ m}$$

We can use the relativistic orbit formula for precession derived earlier. For elliptical orbits, the angular momentum can be related to the semi-major axis via:

$$r_\pm = \frac{l^2}{GM} \cdot \frac{1}{1 \pm e} \quad \Rightarrow \quad l^2 = GM \cdot r_+(1-e)$$

Thus, the perihelion shift per revolution becomes:

$$\delta = \frac{6\pi GM}{c^2 l^2} = \frac{6\pi GM}{c^2 \cdot GM \cdot r_+(1-e)} = \frac{6\pi}{c^2 r_+(1-e)}$$

Plugging in the values:

$$\delta \approx \frac{6\pi}{(3 \times 10^8)^2 \cdot 7 \times 10^{10} \cdot (1-0.2)} \approx 5.0 \times 10^{-7} \text{ radians}$$

Although this is a very small angle per orbit, **the effect is cumulative**. Mercury completes about 415 orbits per century, so the total precession is:

$$\Delta\phi_{\text{century}} = 415 \cdot \delta \approx 2.1 \times 10^{-4} \text{ radians per century}$$

Astronomers measure angular displacement in **arcseconds**, where:

$$1^\circ = 3600'' \quad \Rightarrow \quad 1'' = \frac{2\pi}{360 \cdot 3600} \approx 4.848 \times 10^{-6} \text{ radians}$$

Therefore, the observed perihelion shift is:

$$\frac{2.1 \times 10^{-4}}{4.848 \times 10^{-6}} \approx 43'' \text{ per century}$$

This **matches the observed anomalous precession** of Mercury's orbit, one of the earliest and strongest confirmations of general relativity. Subsequent observations of Venus and Earth's orbits also agree with relativistic predictions.

2.5.1 Newtonian Contributions from Other Planets

The general relativistic contribution of $43''$/century is not the full story — Mercury's observed perihelion shift is roughly $575''$/century. Most of this ($532''$) arises from the gravitational perturbation of the other planets, a purely Newtonian effect.

We approximate the influence of an outer planet of mass M' orbiting at radius R as a circular ring of mass-per-unit-length $M'/(2\pi R)$ in the same plane. Since Mercury orbits much faster, we may average over the outer planet's position:

The Newtonian potential from the Sun is still

$$V_\star(r) = -\frac{GM}{r}$$

and from the ring:

$$V_{\text{ring}}(r) = -\frac{GM'}{2\pi R}\int_0^{2\pi}\frac{d\theta}{\sqrt{R^2+r^2-2Rr\cos\theta}} \approx -\frac{GM'}{R}\left(1+\frac{r^2}{4R^2}+\cdots\right)$$

Dropping r-independent terms, the effective potential per unit mass becomes

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \sum_i \frac{GM_i'}{4R_i^3}r^2$$

where the sum runs over all outer planets.

Switching to $u(\phi) = 1/r$, the perturbed orbit equation is:

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{l^2} = -\alpha u^3$$

where

$$\alpha = \sum_i \frac{G^3M^4}{l^6}\frac{M_i'}{2R_i^3}$$

In the small-α, e limit and to first order in eccentricity, the perihelion shift per orbit is:

$$\delta = 3\pi\,\alpha = 3\pi\sum_i \frac{M_i'}{2M}\left(\frac{r_0}{R_i}\right)^3$$

with $r_0 = (1-e)r_+$.

Using $M = M_\odot \approx 2\times 10^{30}$kg, $r_0 \approx 5.64\times 10^{10}$m, and data:

Planet	$M_i'\,(10^{24}\,\text{kg})$	$R_i\,(10^{11}\,\text{m})$
Venus	4.9	1.08
Earth	6.0	1.52
Mars	0.64	2.28
Jupiter	1900	7.78
Saturn	570	14.0

A quick estimate shows Jupiter and Venus dominate. Summing their contributions yields:

$$\delta \approx 40\times 10^{-7}\,\text{rad/orbit} \quad\Rightarrow\quad 344''\,/\,\text{century}$$

which is larger than the relativistic $43''$ but still short of the $\sim 532''$ from full Newtonian analysis.

Higher-order corrections Including further terms in the ring's potential expansion:

$$V_{\text{ring}}(r) = -2\pi R\frac{GM'}{2\pi R}\left[1 + \frac{r^2}{4R^2} + \frac{9r^4}{64R^4} + \cdots\right]$$

leads to a more accurate perihelion shift formula:

$$\delta = \pi \sum_i \frac{M_i'}{2M}\left[3\left(\frac{r_0}{R_i}\right)^3 + 45\left(\frac{r_0}{R_i}\right)^5 + \cdots\right]$$

Accounting for higher-order terms (especially Venus's significant $(r_0/R) \approx 0.5$) yields:

$$\delta \approx 59 \times 10^{-7}\,\text{rad/orbit} \;\Rightarrow\; 507''\,/\,\text{century}$$

bringing us closer to the full Newtonian prediction of $532''$. With more precise multipole expansions and orbital averaging, one recovers the full $\sim 532''$/century, reconciling observation and Newtonian perturbation theory.

Null Geodesics and the Photon Sphere

We now turn to the motion of massless particles (i.e., light) in the Schwarzschild geometry. As before, we use the general equations of motion derived earlier, but we now impose the **null constraint**:

$$-A(r)c^2\dot{t}^2 + A^{-1}(r)\dot{r}^2 + r^2\dot{\phi}^2 = 0$$

where $A(r) = 1 - \frac{2GM}{c^2 r}$. This replaces the timelike constraint used for massive particles.

Just as in the timelike case, the motion reduces to an effective one-dimensional radial motion:

$$\frac{1}{2}\dot{r}^2 + V_{\text{null}}(r) = \frac{1}{2}\frac{E^2}{c^2}$$

where the **effective potential** is now:

$$V_{\text{null}}(r) = \frac{l^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

This potential behaves differently from the massive case: - As $r \to \infty$, $V_{\text{null}}(r) \to 0^+$ - As $r \to 0$, $V_{\text{null}}(r) \to -\infty$

The potential has a **maximum**, located at:

$$\frac{dV_{\text{null}}}{dr} = 0 \quad \Rightarrow \quad r_\star = \frac{3GM}{c^2}$$

This is known as the **photon sphere**, the radius at which light can orbit the black hole.

The maximum value of the potential is:

$$V_{\text{null}}(r_\star) = \frac{l^2 c^4}{54G^2M^2}$$

This orbit is **unstable**: any small radial perturbation causes the photon to either fall into the black hole or escape to infinity. Nonetheless, this radius plays a central role in black hole optics — light from an accretion disk appears to emerge from near the photon sphere, forming the **bright ring** seen in EHT images (though the resolution may not be sufficient to resolve the photon sphere precisely).

Behavior Based on Energy The fate of a light ray depends on its total energy E and angular momentum l, specifically on the comparison between $E^2/2c^2$ and the barrier height $V_{\text{null}}(r_\star)$:

- **Case 1:** $E < \frac{lc^3}{\sqrt{27}GM}$
  Light with energy below the angular momentum barrier cannot escape from within $r_\star$. It will fall toward the black hole after looping around — or, if coming from outside, will be deflected (scattered) and escape back to infinity.

- **Case 2:** $E > \frac{lc^3}{\sqrt{27}GM}$
  Light now has enough energy to overcome the barrier. If emitted from $r < r_\star$, it may escape to infinity — but only if $r > R_s = \frac{2GM}{c^2}$. Conversely, light from infinity may spiral in and fall into the black hole.

This behavior encodes rich phenomena like gravitational lensing, photon capture cross sections, and the formation of black hole shadows. We will examine light deflection and capture cross sections in more detail in subsequent sections.

Gravitational Lensing

To analyze the deflection of light rays in more detail, we again introduce the inverse radial parameter $u = 1/r$. The null geodesic equation becomes:

$$\left(\frac{du}{d\phi}\right)^2 + u^2\left(1 - \frac{2GM}{c^2}u\right) = \frac{E^2}{l^2c^2}$$

Differentiating this yields the second-order equation:

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2 \tag{1.54}$$

We solve this perturbatively in the small dimensionless parameter $\tilde{\beta} = \frac{GM}{c^2 b}$, where b is the **impact parameter**.

At zeroth order, we ignore the right-hand side:

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0 \quad \Rightarrow \quad u_0 = \frac{1}{b}\sin\phi$$

which describes a straight-line trajectory: $r\sin\phi = b$.

To first order in $\tilde{\beta}$, the equation becomes:

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{3}{b^2}\sin^2\phi = \frac{3}{2b^2}(1 - \cos 2\phi)$$

whose general solution is:

$$u_1 = A\cos\phi + B\sin\phi + \frac{1}{2b^2}(3 + \cos 2\phi)$$

Choosing integration constants $A = \frac{2}{b}$, $B = 0$ to match the asymptotic behavior at $\phi = \pi$, the total solution becomes:

$$u(\phi) = \frac{1}{b}\sin\phi + \frac{GM}{2b^2c^2}(3 + 4\cos\phi + \cos 2\phi)$$

To find the total bending angle, we set $u = 0$ (i.e., $r \to \infty$) and approximate $\sin\phi \approx \phi$, $\cos\phi \approx 1$ near $\phi = 0$. Solving for the deviation, we find:

$$\delta\phi = \frac{4GM}{bc^2} \tag{1.55}$$

This is the celebrated result of **gravitational lensing**: light is deflected by an angle $\delta\phi$ when passing by a massive object at impact parameter b. This prediction, confirmed observationally during the 1919 solar eclipse, was one of the first major triumphs of general relativity.

3 Differential Geometry

3.1 Introduction to Differential Geometry

Describing gravity as a geometric phenomenon requires a mathematical framework capable of handling curved spaces and spacetimes. This framework is provided by differential geometry, which generalizes the tools of calculus to smooth manifolds that are not necessarily flat.

In the context of general relativity, spacetime is modeled as a four-dimensional manifold equipped with a metric tensor that determines distances and causal structure. Unlike Euclidean space, such manifolds can exhibit intrinsic curvature, which is directly related to the distribution of mass-energy.

Differential geometry introduces a range of geometric structures that are necessary for formulating physical theories in curved backgrounds. These include smooth manifolds, charts and atlases for coordinate systems, tangent spaces, vector fields, and tensors. All these objects transform covariantly under changes of coordinates, ensuring that physical laws remain independent of the observer's frame.

This section provides a minimal introduction to the concepts and structures that are essential for formulating general relativity and other geometrically motivated physical theories. The emphasis is on definitions and their logical relationships, with formal proofs and topological details omitted for brevity.

3.2 Manifolds

A manifold is the basic geometric setting for formulating physical theories involving curvature and local smooth structure. Informally, an n-dimensional manifold is a space that, around every point, resembles R^n, although its global structure may be curved or topologically nontrivial.

At this stage, the manifold is assumed to have minimal structure. Notably, no intrinsic notion of distance, angle, or curvature is defined yet. These require additional geometric structures to be introduced later, such as a metric tensor. For now, the focus is on the topological and differentiable properties that define a smooth manifold.

Examples of manifolds include:

- The Euclidean space R^n
- The n-sphere S^n, defined as the set of points in R^{n+1} at unit distance from the origin

- The n-torus $T^n = S^1 \times \cdots \times S^1$

In physics, manifolds arise naturally. Classical configuration spaces, phase spaces, and the state spaces in thermodynamics are all examples of manifolds. In general relativity, spacetime itself is modeled as a smooth four-dimensional manifold, to which additional structure — such as a Lorentzian metric — is applied to describe gravitational dynamics.

The formal definition of a manifold and the associated coordinate systems will follow in subsequent sections. These will provide the language necessary to define vector fields, tensors, and curvature in a consistent and coordinate-independent manner.

3.2.1 Topological Spaces

Before defining manifolds, we begin with the underlying structure of a topological space. This provides the minimal framework necessary for discussing continuity, convergence, and neighbourhoods.

Definition: A *topological space* M is a set equipped with a collection $\mathcal{T}$ of subsets called the topology, satisfying:

(i) $M \in \mathcal{T}$, $\emptyset \in \mathcal{T}$

(ii) The finite intersection of open sets is an open set

(iii) The arbitrary union of open sets is an open set

An open set $O \in \mathcal{T}$ is called a *neighbourhood* of a point $p \in M$ if $p \in O$.

A topological space is called *Hausdorff* if for any two distinct points $p, q \in M$, there exist disjoint neighbourhoods $O_1, O_2 \in \mathcal{T}$ such that $p \in O_1$, $q \in O_2$, and $O_1 \cap O_2 = \emptyset$. This separation property is assumed in all physically meaningful spaces.

Example: The real line R, with open intervals as basic open sets, forms a Hausdorff topological space.

Definition (Homeomorphism): A map $f : M \to \tilde{M}$ between two topological spaces is a *homeomorphism* if:

(i) f is bijective

(ii) f is continuous

(iii) f^{-1} is also continuous

Spaces related by a homeomorphism are topologically equivalent. For example, a torus and a coffee mug are homeomorphic, illustrating that topology preserves qualitative geometric features but not distance or curvature.

3.2.2 Differentiable Manifolds

An n-dimensional differentiable manifold is a Hausdorff topological space M that is locally modeled on R^n and supports smooth coordinate transitions.

Definition: M is a differentiable manifold if:

(i) For every $p \in M$, there exists an open set $O \subset M$ and a homeomorphism $\varphi : O \to U \subset R^n$.

(ii) If $O_\alpha \cap O_\beta \neq \emptyset$, then the transition map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(O_\alpha \cap O_\beta) \to \varphi_\beta(O_\alpha \cap O_\beta)$ is C^∞, as is its inverse.

Each map φ_α is called a *chart*; the collection of charts forms an *atlas*. Coordinates of a point $p \in O_\alpha$ are denoted $\varphi_\alpha(p) = (x^1(p), \ldots, x^n(p))$, or simply x^μ.

If a point lies in overlapping charts, transition functions relate the coordinate descriptions. Compatibility ensures that overlapping charts yield consistent smooth structure.

Multiple compatible atlases can define the same differentiable structure. A differentiable structure is fixed by a maximal atlas — one containing all charts compatible with a given atlas.

3.2.3 Smooth Maps and Diffeomorphisms

The utility of local coordinate charts lies in enabling calculus on manifolds by leveraging familiar analysis on R^n.

A function $f : M \to R$ is said to be *smooth* if for every chart $\varphi : O \subset M \to U \subset R^n$, the composition $f \circ \varphi^{-1} : U \to R$ is smooth.

Given manifolds M and N, a map $f : M \to N$ is smooth if for all charts $\varphi : O \subset M \to U \subset R^{\dim M}$ and $\psi : O' \subset N \to V \subset R^{\dim N}$, the map $\psi \circ f \circ \varphi^{-1} : U \to V$ is smooth.

A *diffeomorphism* is a bijective, smooth map $f : M \to N$ whose inverse f^{-1} is also smooth. If such a map exists, the manifolds M and N are said to be *diffeomorphic*.

Diffeomorphism is a stronger condition than homeomorphism: all diffeomorphic manifolds are homeomorphic, but the converse is not true. Manifolds may admit multiple inequivalent differentiable structures. For instance:

- The 7-sphere S^7 admits exotic smooth structures that are homeomorphic but not diffeomorphic.
- R^n is uniquely smooth for all $n \neq 4$, but R^4 admits uncountably many inequivalent smooth structures.

These exotic structures are mostly of mathematical interest, with limited applications in physics.

3.3 Tangent Spaces

To define differentiation on a manifold M, consider a smooth function $f : M \to R$ and a chart $\varphi : O \subset M \to U \subset R^n$ around a point $p \in M$. This gives a function $f \circ \varphi^{-1} : U \to R$, which we can differentiate using standard calculus on R^n. In coordinates x^μ, we define:

$$\left.\frac{\partial f}{\partial x^\mu}\right|_p := \left.\frac{\partial (f \circ \varphi^{-1})}{\partial x^\mu}\right|_{\varphi(p)}$$

This definition is coordinate-dependent. Our goal is to construct a coordinate-independent notion of differentiation at p, which leads to the concept of the *tangent space* T_pM, the vector space of all derivations at p — linear maps acting on smooth functions f satisfying the Leibniz rule:

$$v(fg) = v(f)g(p) + f(p)v(g) \quad \forall f, g \in C^\infty(M)$$

We will see that any coordinate system $\{x^\mu\}$ around p gives a natural basis $\left\{\frac{\partial}{\partial x^\mu}\right\}_p$ for T_pM, and coordinate transformations act on these basis vectors via the chain rule.

3.3.1 Tangent Vectors

A tangent vector at a point $p \in M$ is defined as a linear map

$$v : C^\infty(M) \to R$$

satisfying the Leibniz rule:

$$v(fg) = v(f)g(p) + f(p)v(g)$$

for all smooth functions $f, g \in C^\infty(M)$. The set of all such derivations at p forms a real vector space, called the *tangent space* at p, denoted T_pM.

Given a chart $(x^1, \ldots, x^n)$ around p, we define the basis of T_pM as:

$$\left\{ \left. \frac{\partial}{\partial x^\mu} \right|_p \right\}_{\mu=1}^n$$

Each basis vector acts on a function $f \in C^\infty(M)$ via:

$$\left. \frac{\partial}{\partial x^\mu} \right|_p (f) := \left. \frac{\partial (f \circ \varphi^{-1})}{\partial x^\mu} \right|_{\varphi(p)}$$

Any tangent vector $v \in T_pM$ can be expressed as:

$$v = v^\mu \left. \frac{\partial}{\partial x^\mu} \right|_p$$

with components $v^\mu \in R$. Under a change of coordinates, these components transform via the Jacobian:

$$v'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} v^\mu$$

This makes tangent vectors geometric objects independent of coordinates.

Theorem. The tangent space T_pM at a point $p \in M$ is an n-dimensional real vector space. Given a coordinate chart $(x^1, \ldots, x^n)$ around p, the vectors

$$\left\{ \left. \frac{\partial}{\partial x^\mu} \right|_p \right\}_{\mu=1}^n$$

form a basis for T_pM.

Any tangent vector $X_p \in T_pM$ can be expressed as

$$X_p = X^\mu \left. \frac{\partial}{\partial x^\mu} \right|_p$$

where $X^\mu \in R$ are the components of the vector in the chosen coordinate basis.

3.3.2 Changing Coordinates

Tangent vectors are geometric objects that exist independently of any coordinate system. However, to express them concretely, we often introduce a chart φ with coordinates x^μ, which induces a natural basis for the tangent space T_pM given by:

$$\left\{ \left. \frac{\partial}{\partial x^\mu} \right|_p \right\}$$

This is known as a *coordinate basis*, since each basis vector corresponds to a partial derivative with respect to a coordinate direction.

Under a change of coordinates $x^\mu \to x'^\nu$, the basis vectors transform as:

$$\left. \frac{\partial}{\partial x'^\nu} \right|_p = \frac{\partial x^\mu}{\partial x'^\nu} \left. \frac{\partial}{\partial x^\mu} \right|_p$$

and hence the components X^μ of any vector $X_p = X^\mu \frac{\partial}{\partial x^\mu}$ must transform accordingly to preserve the geometric object:

$$X'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} X^\mu$$

This ensures that the vector itself is invariant:

$$X_p = X^\mu \frac{\partial}{\partial x^\mu} = X'^\nu \frac{\partial}{\partial x'^\nu}$$

Coordinate vs. Non-Coordinate Bases. While coordinate bases arise naturally from charts, we can also work with more general bases $\{e_\mu\}$, not necessarily related to coordinate derivatives. These are called *non-coordinate bases*. A particularly important example of this is the *vielbein* or *frame field*, which provides a local orthonormal basis at each point and will be discussed in Section 3.4.

In all cases, physical and geometric quantities must be independent of the coordinate system chosen. This principle will guide our construction of all further structures on manifolds.

3.4 Vector Fields

Previously, we defined tangent vectors at a single point $p \in M$. It is often more useful to consider a smooth assignment of a tangent vector to every point in the manifold. Such an object is called a *vector field*.

A vector field X on a manifold M is a smooth map that assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$. Equivalently, a vector field is a linear map:

$$X : C^\infty(M) \to C^\infty(M)$$

such that $X(f)(p) = X_p(f)$, where X_p acts as a derivation on $f \in C^\infty(M)$.

The space of all smooth vector fields on M is denoted by $X(M)$.

In a coordinate chart (x^μ), any vector field can be written locally as:

$$X = X^\mu(x) \frac{\partial}{\partial x^\mu}$$

where the $X^\mu \in C^\infty(O)$ are smooth functions defined on the open set $O \subset M$ covered by the chart. To define X globally, such local expressions must be smoothly compatible on chart overlaps.

3.4.1 Integral Curves

An alternative perspective on vector fields is through the notion of flows. A *flow* on a manifold M is a one-parameter family of diffeomorphisms $\sigma_t : M \to M$, satisfying

$$\sigma_0 = \mathrm{id}_M, \quad \sigma_s \circ \sigma_t = \sigma_{s+t}, \quad \text{and} \quad \sigma_{-t} = \sigma_t^{-1}$$

This defines a smooth action of R on M, with each point $p \in M$ tracing out a smooth curve under the flow. These curves are called *integral curves* or *orbits* of the flow.

The vector field X generated by the flow is defined as the tangent to these integral curves:

$$X^\mu(x^\mu(t)) = \frac{dx^\mu(t)}{dt}$$

where $x^\mu(t)$ are the coordinate functions of the flow line passing through a point.

Conversely, given a smooth vector field X on M, we can construct its integral curves by solving the ordinary differential equation

$$\frac{dx^\mu}{dt} = X^\mu(x(t)), \quad x^\mu(0) = x^\mu_{\text{initial}}$$

This equation defines a curve whose velocity at each point equals the vector field evaluated at that point. These curves are often visualized as streamlines of the vector field, and together they form the *flow* generated by X, at least locally.

The existence and uniqueness of such flows is guaranteed by standard results in differential equations, provided X is smooth.

3.4.2 Lie Derivatives, Push-Forward, and Pull-Back

Push-Forward. Given a smooth map $f : M \to N$ between manifolds, the *push-forward* (or *differential*) f_* maps tangent vectors from T_pM to $T_{f(p)}N$. For $X_p \in T_pM$, the push-forward is defined via its action on smooth functions $g \in C^\infty(N)$ as:

$$(f_* X_p)(g) := X_p(g \circ f)$$

This constructs a vector at $f(p)$ in N from a vector at p in M. Intuitively, it transports directions forward along the map f.

Pull-Back. In contrast, the *pull-back* operates in the opposite direction. Given $f : M \to N$ and a differential form $\omega \in \Omega^k(N)$, the pull-back $f^*\omega \in \Omega^k(M)$ is the unique form on M satisfying:

$$(f^*\omega)_p(v_1, \ldots, v_k) := \omega_{f(p)}(f_* v_1, \ldots, f_* v_k)$$

for $v_i \in T_pM$. That is, we use f_* to push vectors forward, then apply the original form ω.

Lie Derivative. The Lie derivative measures how a tensor field changes as it flows along a vector field X. For a scalar function f, the Lie derivative is just the directional derivative:

$$\mathcal{L}_X f = X(f)$$

For a vector field Y, the Lie derivative is given by the commutator:

$$\mathcal{L}_X Y = [X, Y]$$

For a differential form ω, the Lie derivative obeys Cartan's formula:

$$\mathcal{L}_X \omega = d(i_X \omega) + i_X(d\omega)$$

Here, i_X denotes the interior product (contraction) with X, and d is the exterior derivative.

The Lie derivative preserves the geometric meaning of a field under flow, and is coordinate-independent. It plays a central role in symmetries, conservation laws, and the formulation of physical theories on manifolds.

3.5 Tensors

Let V be a vector space. The *dual space* V^* is defined as the set of all linear maps from $V \to R$. That is,

$$V^* := \{\omega : V \to R \mid \omega \text{ is linear}\}.$$

This mirrors the familiar bra-ket notation from quantum mechanics, where bras $\langle\phi| \in \mathcal{H}^*$ act on kets $|\psi\rangle \in \mathcal{H}$ via $\langle\phi|\psi\rangle \in C$.

Given a basis $\{e_\mu\}$ for V, the *dual basis* $\{f^\mu\}$ for V^* is defined by

$$f^\nu(e_\mu) = \delta^\nu_\mu.$$

A vector $X \in V$ can be expressed as $X = X^\mu e_\mu$, and its action under $f^\nu \in V^*$ gives $f^\nu(X) = X^\nu$.

Although $V \cong V^*$ in finite dimensions, this identification is basis dependent. In contrast, the double dual $(V^*)^*$ is canonically isomorphic to V in a natural, basis-independent way.

A *tensor* is a multilinear map involving both vectors and covectors. Formally, a tensor of type (k, ℓ) is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \times \underbrace{V \times \cdots \times V}_{\ell \text{ times}} \to R.$$

The space of such tensors is denoted $T^k_\ell(V)$.

Tensors transform under changes of basis in a well-defined way and include familiar objects such as: - Scalars: type $(0, 0)$ - Vectors: type $(1, 0)$ - Covectors (dual vectors): type $(0, 1)$ - Linear maps (e.g., matrices): type $(1, 1)$ - Bilinear forms: type $(0, 2)$

Given a basis $\{e_\mu\}$ for V and dual basis $\{f^\nu\}$, any tensor $T \in T^k_\ell(V)$ can be expressed as

$$T = T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_\ell}\, e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \otimes f^{\nu_1} \otimes \cdots \otimes f^{\nu_\ell},$$

where the coefficients $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_\ell} \in R$ specify the tensor in components.

This framework extends naturally to tensors on manifolds, where each tangent space $T_p(M)$ and cotangent space $T^*_p(M)$ give rise to tensor fields defined pointwise over the manifold.

3.5.1 Covectors and One-Forms

At each point $p \in M$, the cotangent space $T_p^*(M)$ is the dual space to the tangent space $T_p(M)$. Elements of $T_p^*(M)$ are called *covectors* or *one-forms*. Given a basis $\{e_\mu\}$ for $T_p(M)$, the dual basis $\{dx^\mu\}$ satisfies

$$dx^\mu(e_\nu) = \delta^\mu_\nu.$$

Any one-form $\omega \in T_p^*(M)$ can be written as $\omega = \omega_\mu dx^\mu$.

A *one-form field* assigns a covector to every point $p \in M$ smoothly. The collection of all one-forms on M is denoted $\Lambda^1(M)$.

A canonical example comes from the differential of a smooth function $f \in C^\infty(M)$, defined by

$$df(X) := X(f),$$

for any vector field X. In coordinates, this takes the form

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu.$$

Under a change of coordinates $x^\mu \to \tilde{x}^\nu$, one-forms transform *covariantly*:

$$dx^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} d\tilde{x}^\nu, \quad \text{so that} \quad \omega = \omega_\mu dx^\mu = \tilde{\omega}_\nu d\tilde{x}^\nu,$$

with

$$\tilde{\omega}_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \omega_\mu.$$

This covariant transformation is the dual behavior to the contravariant transformation of vectors. The index placement in ω_μ (subscript) reflects this covariance.

Lie Derivatives and Pull-Backs of One-Forms

Just as vector fields can be *pushed forward* under a smooth map $\phi : M \to N$, one-forms behave oppositely: they are *pulled back*. Given a one-form $\omega \in \Lambda^1(N)$, the pull-back $\phi^*\omega \in \Lambda^1(M)$ is defined by

$$(\phi^*\omega)(X) = \omega(\phi_* X)$$

for any vector field $X \in X(M)$. In coordinates, with x^μ on M and y^α on N, we have

$$(\phi^*\omega)_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}.$$

To define the **Lie derivative** of a one-form ω along a vector field X, we use the flow σ_t generated by X. The Lie derivative is then:

$$\mathcal{L}_X \omega = \lim_{t \to 0} \frac{(\sigma_t^* \omega)_p - \omega_p}{t}.$$

Using the expansion of $\sigma_t^* dx^\mu$, we find:

$$\mathcal{L}_X(dx^\mu) = \frac{\partial X^\mu}{\partial x^\nu} dx^\nu,$$

and for a general one-form $\omega = \omega_\mu dx^\mu$, the Lie derivative becomes:

$$\mathcal{L}_X \omega = (X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu)\, dx^\mu.$$

Tensors and Tensor Fields

On a manifold M, we already have the tangent space $T_p(M)$ and its dual, the cotangent space $T_p^*(M)$ at each point $p \in M$. A **tensor** at p is a multilinear map that eats vectors and covectors and spits out real numbers.

More precisely, a (k, ℓ)-tensor at p is a multilinear map

$$T : \underbrace{T_p^*(M) \times \cdots \times T_p^*(M)}_{k \text{ times}} \times \underbrace{T_p(M) \times \cdots \times T_p(M)}_{\ell \text{ times}} \to R.$$

This object is linear in each slot. Tensors of type $(1, 0)$ are vectors, and $(0, 1)$ are one-forms. A $(0, 2)$-tensor is a bilinear form, while a $(1, 1)$-tensor acts like a linear map from vectors to vectors.

In a local coordinate system $\{x^\mu\}$, we define a basis of vector fields $\left\{\frac{\partial}{\partial x^\mu}\right\}$ and one-forms $\{dx^\mu\}$. A (k, ℓ)-tensor T can then be expanded as

$$T = T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_\ell} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_\ell},$$

where $T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_\ell}$ are smooth functions on M called the components of the tensor.

A **tensor field** is a smooth assignment of such a tensor to each point on M. The set of all (k, ℓ)-tensor fields is denoted $\mathcal{T}^k_\ell(M)$.

Under coordinate transformations, tensor components transform via:

$$T^{\mu_1' \ldots \mu_k'}_{\nu_1' \ldots \nu_\ell'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_\ell}}{\partial x^{\nu_\ell'}} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_\ell}.$$

This ensures that the full tensor expression is coordinate-independent even though its components change.

Tensors are the backbone of general relativity, where key physical quantities like the metric, energy-momentum tensor, and curvature are all tensor fields.

The Identity Tensor. Every manifold comes equipped with a natural $(1, 1)$-tensor field δ, known as the identity tensor. It acts by pairing a one-form $\omega \in T_p^*(M)$ and a vector $X \in T_p(M)$ to return a real number:

$$\delta(\omega, X) = \omega(X).$$

In a coordinate basis, this corresponds to

$$\delta(f^\mu, e_\nu) = f^\mu(e_\nu) = \delta^\mu_\nu,$$

where $f^\mu = dx^\mu$ and $e_\nu = \partial/\partial x^\nu$. This is simply the Kronecker delta, enforcing the duality between vectors and covectors.

Operations on Tensor Fields

There are several standard operations that can be performed on tensor fields. Below we list the most important ones along with their coordinate expressions.

1. Tensor Product. Given a (k,ℓ) tensor field T and a (m,n) tensor field S, their tensor product $T\otimes S$ is a $(k+m,\ell+n)$ tensor field defined by

$$(T\otimes S)(\omega_1,\ldots,\omega_{k+m},X_1,\ldots,X_{\ell+n}) = T(\omega_1,\ldots,\omega_k,X_1,\ldots,X_\ell)\cdot S(\omega_{k+1},\ldots,\omega_{k+m},X_{\ell+1},\ldots,X_{\ell+n}).$$

In local coordinates, if $T = T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_\ell}$ and $S = S^{\alpha_1\ldots\alpha_m}_{\beta_1\ldots\beta_n}$, then

$$(T\otimes S)^{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_m}_{\nu_1\ldots\nu_\ell\beta_1\ldots\beta_n} = T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_\ell}\cdot S^{\alpha_1\ldots\alpha_m}_{\beta_1\ldots\beta_n}.$$

2. Contraction. Contraction reduces the valence of a tensor by contracting one upper and one lower index. For example, given a $(1,1)$ tensor T, the trace is the scalar

$$\mathrm{Tr}(T) = T^\mu_\mu.$$

In general, for a (k,ℓ) tensor $T^{\mu_1\cdots\mu_k}_{\nu_1\ldots\nu_\ell}$, contraction over μ_i and ν_j gives a $(k-1,\ell-1)$ tensor.

3. Pull-back and Push-forward. Let $\phi : M \to N$ be a smooth map between manifolds.

Push-forward of a vector field:

$$\phi_* X_p = \left(\frac{\partial y^\alpha}{\partial x^\mu}X^\mu\right)\frac{\partial}{\partial y^\alpha}.$$

Pull-back of a one-form:

$$(\phi^*\omega)_p = \left(\frac{\partial x^\mu}{\partial y^\alpha}\omega_\alpha\right)dx^\mu.$$

These operations extend naturally to higher-rank tensors by linearly acting on each index.

4. Lie Derivative. The Lie derivative $\mathcal{L}_X T$ of a tensor field T with respect to a vector field X measures the change of T along the flow generated by X.

Example: If $\omega = \omega_\mu dx^\mu$ is a one-form, then

$$\mathcal{L}_X\omega = X^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu X^\nu \quad\Rightarrow\quad (\mathcal{L}_X\omega)_\mu = X^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu X^\nu.$$

For a vector field $Y = Y^\mu\partial_\mu$,

$$[\mathcal{L}_X Y]^\mu = X^\nu\partial_\nu Y^\mu - Y^\nu\partial_\nu X^\mu.$$

5. Index Raising and Lowering. Once a metric $g_{\mu\nu}$ is introduced (in Section 3), we can raise or lower indices. For instance:

$$Y_\mu = g_{\mu\nu}Y^\nu, \qquad Y^\mu = g^{\mu\nu}Y_\nu.$$

6. Symmetrization and Antisymmetrization. Given a tensor $T_{\mu\nu}$, we can define:

$$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \quad \text{(Symmetric part)},$$
$$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \quad \text{(Antisymmetric part)}.$$

These operations are fundamental in defining objects such as the electromagnetic field tensor, Riemann curvature, and differential forms.

These operations allow us to construct, transform, and interpret tensorial quantities on a manifold in a coordinate-invariant way.

3.6 Differential Forms

Differential forms provide a general framework for integration on manifolds and encode geometrical and physical data in a coordinate-independent way. They are antisymmetric tensor fields built from the cotangent bundle.

3.6.1 Definition

A **differential k-form** on a smooth manifold M is a totally antisymmetric $(0, k)$ tensor field. That is,

$$\omega \in \Lambda^k(M) := \Gamma\left(\bigwedge^k T^*M\right).$$

Locally, in a chart (x^μ), a differential k-form can be written as

$$\omega = \frac{1}{k!}\omega_{\mu_1\ldots\mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k},$$

where the components $\omega_{\mu_1\ldots\mu_k}$ are smooth and totally antisymmetric.

3.6.2 Wedge Product

The wedge product $\wedge$ is an associative, bilinear, and antisymmetric product on forms:

$$\omega \in \Lambda^k(M), \quad \eta \in \Lambda^l(M) \quad \Rightarrow \quad \omega \wedge \eta \in \Lambda^{k+l}(M),$$

satisfying

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega.$$

In coordinates, this product combines forms with antisymmetrized tensor products of the dx^μ.

3.6.3 Exterior Derivative

The exterior derivative $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ satisfies:

- $d(f) = df$ for $f \in C^\infty(M)$,
- $d^2 = 0$,
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for $\omega \in \Lambda^k(M)$.

The operator d gives the de Rham complex and encodes the notion of differential without reliance on a metric.

3.6.4 Closed and Exact Forms

A form ω is:

- **Closed** if $d\omega = 0$,
- **Exact** if $\omega = d\alpha$ for some α.

Every exact form is closed (since $d^2 = 0$), but not every closed form is exact. This leads to the de Rham cohomology:

$$H^k_{\mathrm{dR}}(M) := \frac{\ker(d : \Lambda^k \to \Lambda^{k+1})}{\mathrm{im}(d : \Lambda^{k-1} \to \Lambda^k)}.$$

3.6.5 Pullback of Forms

Given a smooth map $\phi : M \to N$ and a form $\omega \in \Lambda^k(N)$, the **pullback** $\phi^*\omega \in \Lambda^k(M)$ is defined such that

$$(\phi^*\omega)(X_1, \ldots, X_k) = \omega(\phi_* X_1, \ldots, \phi_* X_k).$$

In coordinates, the pullback acts as:

$$(\phi^*\omega)_\mu = \frac{\partial y^\alpha}{\partial x^\mu}\omega_\alpha.$$

3.6.6 Applications in Physics

Differential forms appear widely in modern theoretical physics:

- The electromagnetic field strength is a closed 2-form $F = dA$.
- Conservation of charge: $d{\star}J = 0$, where J is the current 1-form and $\star$ is the Hodge dual.
- General Relativity uses differential forms to express curvature and volume integrals in coordinate-free language.

3.6.7 Differential Forms in Electromagnetism and Thermodynamics

Differential forms provide a natural language for expressing fundamental laws of physics in a coordinate-free and geometrically meaningful way.

Electromagnetism: The electromagnetic potential is encoded in a 1-form $A \in \Lambda^1(M)$, from which the field strength 2-form is constructed:

$$F = dA \in \Lambda^2(M)$$

This automatically satisfies the homogeneous Maxwell equations:

$$dF = 0$$

The inhomogeneous Maxwell equations are written in terms of the Hodge dual:

$$d \star F = \star J$$

where $J \in \Lambda^1(M)$ is the current 1-form and $\star$ denotes the Hodge star operator, defined via the spacetime metric.

Thermodynamics: In thermodynamics, differential forms can describe the structure of state spaces and thermodynamic processes.

A common example is the first law of thermodynamics, which can be encoded as a 1-form:

$$\theta = dU - TdS + PdV$$

where $\theta \in \Lambda^1(M)$ represents the infinitesimal heat/work exchange, U is internal energy, T is temperature, S is entropy, and P, V are pressure and volume. The condition for a thermodynamic system to be in equilibrium corresponds to the integrability condition $\theta = 0$.

Moreover, contact geometry generalizes this setup using a contact 1-form on a thermodynamic phase space, allowing the formulation of Legendre transforms and the thermodynamic potential hierarchy.

Differential forms thus serve as powerful tools in expressing physical laws with clarity and geometric insight.

3.7 Integration on Manifolds

Integration on manifolds generalizes the familiar notions of line, surface, and volume integrals from calculus. To define integration in a coordinate-free way, we rely on differential forms.

3.7.1 Integrating over Submanifolds

Let $\omega \in \Lambda^k(M)$ be a differential k-form, and let $S \subset M$ be an oriented k-dimensional submanifold. The integral of ω over S is defined as:

$$\int_S \omega$$

This is a natural generalization of vector calculus: 1-forms integrate along curves, 2-forms over surfaces, and so on. The orientation of S is necessary for the sign of the integral to be well-defined.

3.7.2 Stokes' Theorem

Stokes' Theorem provides a unifying framework for many classical theorems such as the fundamental theorem of calculus, Green's theorem, Gauss' divergence theorem, and the classical Stokes' theorem. It states:

$$\int_S d\omega = \int_{\partial S} \omega$$

Here S is an oriented $(k+1)$-dimensional submanifold, $\omega \in \Lambda^k(M)$, and ∂S denotes the boundary of S, which is naturally oriented.

This theorem is valid in any dimension and does not require coordinates. It is truly the cornerstone of differential geometry and topology.

3.7.3 The Mother of All Integral Theorems

Stokes' Theorem is often called "The Mother of All Integral Theorems" because it generalizes:

- **Fundamental Theorem of Calculus:** $\int_a^b f'(x)dx = f(b) - f(a)$
- **Green's Theorem:** $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{\partial D} Pdx + Qdy$
- **Divergence Theorem:** $\iiint_V (\nabla \cdot \vec{F})\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$
- **Classical Stokes' Theorem:** $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$

All of these are special cases of the single, elegant principle:

$$\boxed{\int_S d\omega = \int_{\partial S} \omega}$$

This formulation underscores the deep relationship between differentiation (via d) and integration, and is one of the most profound ideas in modern geometry and physics.

4 Riemannian Geometry and the Metric

To measure lengths, angles, and volumes on a manifold, we introduce a new structure: a **metric**. This turns our differentiable manifold into a Riemannian manifold.

4.1 The Metric Tensor

A **Riemannian metric** g on a manifold M is a smooth assignment of an inner product

$$g_p : T_pM \times T_pM \to R$$

at each point $p \in M$, such that:

- g_p is bilinear and symmetric.
- $g_p(X, X) > 0$ for all non-zero $X \in T_pM$ (positive-definite).

In local coordinates $\{x^\mu\}$, the metric is written as a symmetric 2-tensor:

$$g = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu$$

The components $g_{\mu\nu}$ define a symmetric matrix that varies smoothly across the manifold.

4.2 Measuring with the Metric

Once we have a metric, we can compute:

- **Lengths** of tangent vectors: $\|X\| = \sqrt{g(X, X)}$
- **Angles** between vectors: $\cos\theta = \dfrac{g(X, Y)}{\|X\| \cdot \|Y\|}$

- **Lengths of curves:** For a smooth curve $\gamma(t)$, $a \leq t \leq b$,

$$\text{Length}(\gamma) = \int_a^b \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} \, dt$$

- **Volumes** via the volume form:

$$dV = \sqrt{\det g} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

4.2.1 Properties of the Metric

A **metric** g is a $(0, 2)$ tensor field on a manifold M that allows us to define lengths and angles. It satisfies:

- **Symmetry:** $g(X, Y) = g(Y, X)$ for all $X, Y \in T_pM$
- **Non-degeneracy:** If $g(X, Y) = 0$ for all Y, then $X = 0$

In local coordinates, the metric takes the form

$$g = g_{\mu\nu}(x) \, dx^\mu \otimes dx^\nu$$

which is often abbreviated via the **line element**

$$ds^2 = g_{\mu\nu}(x) \, dx^\mu dx^\nu$$

The components $g_{\mu\nu}$ can be extracted by evaluating the metric on basis vectors:

$$g_{\mu\nu}(x) = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$$

The metric matrix $g_{\mu\nu}$ is symmetric and, by Sylvester's law of inertia, it can always be diagonalized at a point with a fixed number of positive and negative eigenvalues. This leads to the definition of the **signature** of the metric:

- **Riemannian:** All positive (e.g., $(+++)$)
- **Lorentzian:** One negative, rest positive (e.g., $(-+++)$)

4.2.2 Riemannian Manifolds

A **Riemannian manifold** is a smooth manifold M equipped with a metric g that is:

- **Positive-definite:** $g(X, X) > 0$ for all non-zero $X \in T_pM$

This condition ensures that all distances and angles measured by g are real and non-negative. The metric defines:

- **Length of a vector:** $\|X\| = \sqrt{g(X, X)}$
- **Angle between vectors:** via the inner product $g(X, Y)$

- **Length of a curve:** For a path $\gamma(t)$, the length is

$$L[\gamma] = \int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt$$

- **Geodesic distance:** The distance between points $p, q \in M$ is the infimum of $L[\gamma]$ over all paths γ from p to q

In contrast, a **Lorentzian manifold** allows $g(X, X)$ to be negative for some X, as in general relativity. Unless otherwise stated, we take all metrics to be Riemannian from here on.

4.2.3 Lorentzian Manifolds

A **Lorentzian manifold** is a smooth manifold M equipped with a metric g of signature $(-, +, +, +, \ldots)$. Unlike Riemannian metrics, the Lorentzian metric is *not* positive-definite.

This means for a tangent vector $X \in T_pM$, the norm $g(X, X)$ can be:

- **Timelike** if $g(X, X) < 0$
- **Null (or lightlike)** if $g(X, X) = 0$ and $X \neq 0$
- **Spacelike** if $g(X, X) > 0$

This classification allows us to model causal structure and the behavior of light and matter in spacetime. In particular:

- Lorentzian manifolds provide the geometric framework of **general relativity**.
- Timelike and null vectors define allowable worldlines for massive and massless particles, respectively.

The standard model of 4-dimensional spacetime in general relativity is a 4-dimensional Lorentzian manifold with signature $(-, +, +, +)$.

The Metric as an Isomorphism

The metric g defines a natural isomorphism between the tangent and cotangent spaces of a manifold M. This allows us to convert vectors to covectors and vice versa using the metric components.

Lowering an Index (Vector to Covector): Given a vector field $X = X^\mu \partial_\mu$, we can associate to it a one-form (also called a covector) via the metric:

$$X_\nu = g_{\mu\nu} X^\mu \qquad \Rightarrow \qquad X^\flat = X_\nu\, dx^\nu$$

This operation is sometimes called the *flat* map, or the musical isomorphism $\flat$.

Raising an Index (Covector to Vector): Conversely, given a one-form $\omega = \omega_\mu \, dx^\mu$, we can associate a vector field using the inverse metric $g^{\mu\nu}$:

$$\omega^\mu = g^{\mu\nu}\omega_\nu \qquad \Rightarrow \qquad \omega^\sharp = \omega^\mu \, \partial_\mu$$

This operation is called the *sharp* map, denoted $\sharp$.

Non-Degeneracy: These maps are only possible because the metric g is non-degenerate, meaning $g_{\mu\nu}$ has an inverse $g^{\mu\nu}$. This guarantees a one-to-one correspondence between vectors and covectors.

The Volume Form

Once a metric g is introduced on a smooth n-dimensional manifold M, it defines a natural volume form — a top-degree differential form that allows us to measure volume.

Let $\{x^1, \dots, x^n\}$ be local coordinates and the metric written as

$$g = g_{\mu\nu}(x) \, dx^\mu \otimes dx^\nu$$

Then the volume form is given by

$$\boxed{\mathrm{vol}_g = \sqrt{|\det g|} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n}$$

where:

- $\det g$ is the determinant of the matrix $[g_{\mu\nu}]$,
- The square root $\sqrt{|\det g|}$ ensures correct transformation under coordinate changes.

This volume form allows us to integrate scalar functions $f \in C^\infty(M)$ over M as:

$$\int_M f \, \mathrm{vol}_g$$

4.2.4 The Hodge Dual

Given a Riemannian or Lorentzian manifold (M, g) of dimension n, the **Hodge dual** is an isomorphism

$$\star : \Lambda^p(M) \to \Lambda^{n-p}(M)$$

that maps a p-form to an $(n-p)$-form. It depends on both the metric g and the orientation of M.

Let $\{\theta^1, \dots, \theta^n\}$ be an oriented orthonormal coframe at a point $p \in M$. Then the Hodge dual of a simple p-form

$$\omega = \theta^{i_1} \wedge \cdots \wedge \theta^{i_p}$$

is given by

$$\star\omega = \frac{1}{(n-p)!} \epsilon^{i_1 \dots i_p}{}_{j_{p+1} \dots j_n} \theta^{j_{p+1}} \wedge \cdots \wedge \theta^{j_n}$$

where ϵ is the totally antisymmetric Levi-Civita symbol and indices are raised and lowered using the metric g.

In local coordinates, for a general p-form

$$\omega = \frac{1}{p!}\omega_{\mu_1\dots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

the Hodge dual is

$$\star\omega = \frac{\sqrt{|\det g|}}{p!(n-p)!}\omega^{\mu_1\dots\mu_p}\epsilon_{\mu_1\dots\mu_p\nu_1\dots\nu_{n-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}}$$

Properties:

- $\star\star\,\omega = (-1)^{p(n-p)}\omega$ for Riemannian metrics.
- The Hodge star allows one to define an inner product on forms:

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \star\eta$$

- The Hodge star is central in defining codifferentials and Laplace operators on differential forms.

4.2.5 A Sniff of Hodge Theory and Hodge's Theorem

Let (M, g) be a compact, oriented Riemannian manifold. The Hodge star operator $\star$ and the exterior derivative d allow us to define the **codifferential** δ by

$$\delta := (-1)^{np+n+1} \star d\star : \Lambda^p(M) \to \Lambda^{p-1}(M)$$

Using this, we define the **Laplace–de Rham operator** as

$$\Delta := d\delta + \delta d$$

A p-form $\omega \in \Lambda^p(M)$ is called **harmonic** if

$$\Delta\omega = 0$$

The set of all harmonic p-forms on M is denoted

$$\mathcal{H}^p(M) := \{\omega \in \Lambda^p(M) \mid \Delta\omega = 0\}$$

Hodge's Theorem. On any compact, oriented Riemannian manifold M, there is an isomorphism:

$$\mathcal{H}^p(M) \cong H^p_{\mathrm{dR}}(M)$$

where $H^p_{\mathrm{dR}}(M)$ is the p-th de Rham cohomology group.

This result implies that every cohomology class has a unique harmonic representative. Moreover, the **Betti numbers** b_p, which are defined as

$$b_p := \dim H^p_{\mathrm{dR}}(M)$$

can equivalently be computed as the number of linearly independent harmonic p-forms:

$$b_p = \dim \mathcal{H}^p(M)$$

Hodge theory thus provides a powerful link between differential geometry, analysis, and topology.

4.3 Connections and Curvature

Vector fields act as differential operators on smooth functions, giving us a natural notion of differentiation: $X(f)$. However, for general tensor fields, differentiating is more subtle. Tensors at different points live in distinct vector spaces, so they cannot be directly subtracted or compared.

To make sense of differentiation on a manifold, we introduce a new structure called a **connection**. A connection defines a rule for comparing vectors at nearby points and allows us to differentiate tensor fields in a way that respects both smoothness and geometry.

This leads to the concept of the **covariant derivative**. Given vector fields X and Y, the covariant derivative of Y along X is denoted

$$\nabla_X Y$$

and satisfies the following key properties:

- Linearity in both X and Y.
- Leibniz rule: $\nabla_X(fY) = X(f)Y + f\nabla_X Y$ for $f \in C^\infty(M)$.
- Tensorial in X: $\nabla_{fX} Y = f\nabla_X Y$.

The covariant derivative generalizes directional derivatives from flat space to curved manifolds. It is a crucial tool in defining curvature and geodesics, and is foundational to Riemannian geometry and general relativity.

4.3.1 The Covariant Derivative

The covariant derivative provides a way to differentiate tensor fields on a manifold in a manner that respects the manifold's geometric structure. Unlike partial derivatives, which depend on a coordinate chart, the covariant derivative incorporates additional geometric data — a *connection* — to ensure that the derivative is well-defined under changes of coordinates.

Given a vector field X and a tensor field T, the covariant derivative of T along X is denoted $\nabla_X T$. The operation ∇ takes as input a vector field and a tensor field and returns another tensor field of the same type as T.

For vector fields X, Y, the covariant derivative $\nabla_X Y$ is again a vector field, and it must satisfy the following axioms:

1. **Linearity in the direction vector field:**

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z \quad \text{for } f, g \in C^\infty(M)$$

2. **Leibniz rule (product rule):**

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y \quad \text{for } f \in C^\infty(M)$$

3. **Tensoriality in the lower slot:**

$$\nabla_X f = X(f)$$

for $f \in C^\infty(M)$, recovering the usual derivative of a function.

In Coordinates: Let $\{x^\mu\}$ be a local coordinate system and $\left\{\frac{\partial}{\partial x^\mu}\right\}$ the associated coordinate basis. The covariant derivative of the basis vector fields is given by

$$\nabla_\nu \left(\frac{\partial}{\partial x^\mu}\right) = \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda}$$

where $\Gamma^\lambda_{\nu\mu}$ are the **connection coefficients** or **Christoffel symbols**.

Given a vector field $V = V^\mu \frac{\partial}{\partial x^\mu}$, the covariant derivative in the direction ∂_ν is:

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\rho} V^\rho$$

This shows that the covariant derivative differs from the partial derivative by an extra term involving the Christoffel symbols, which account for the curvature or twisting of the manifold.

Covariant Derivative of General Tensors: For a (k,l)-tensor field $T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}$, the covariant derivative is defined as:

$$\begin{aligned}\nabla_\rho T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} = \partial_\rho T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} &+ \sum_{i=1}^{k} \Gamma^{\mu_i}_{\rho\lambda} T^{\mu_1\ldots\lambda\ldots\mu_k}_{\nu_1\ldots\nu_l} \\ &- \sum_{j=1}^{l} \Gamma^{\lambda}_{\rho\nu_j} T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\lambda\ldots\nu_l}\end{aligned}$$

Each upper index gets a $+$ Christoffel term, and each lower index gets a $-$ Christoffel term. This structure ensures that ∇ preserves tensorial transformation rules.

Interpretation: The covariant derivative can be viewed as the rate of change of a tensor field with respect to a given vector direction, while taking into account the underlying geometry of the manifold. This operation allows us to define concepts like parallel transport, geodesics, curvature, and more.

Remark: On a manifold equipped with a metric $g_{\mu\nu}$, there exists a unique connection — the Levi-Civita connection — that is compatible with the metric (i.e., $\nabla g = 0$) and torsion-free. This is the connection used in Riemannian and Lorentzian geometry.

4.3.2 The Connection is Not a Tensor

Although the covariant derivative involves objects called connection coefficients (or Christoffel symbols), it is important to note that these do *not* transform as tensors under coordinate changes.

To see this, consider the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ defined through the action of the covariant derivative on a vector field V^μ:

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\rho} V^\rho$$

If $\Gamma^\mu_{\nu\rho}$ were the components of a $(1,2)$-tensor, then under a coordinate transformation $x^\mu \mapsto \tilde{x}^\alpha$, they would obey the usual tensorial transformation law:

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} \frac{\partial x^\rho}{\partial \tilde{x}^\gamma} \Gamma^\mu_{\nu\rho}$$

However, this is **not** the transformation law that $\Gamma^{\mu}_{\nu\rho}$ follows. Instead, under a change of coordinates, the Christoffel symbols pick up an additional inhomogeneous term:

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\gamma}} \Gamma^{\mu}_{\nu\rho} + \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial \tilde{x}^{\beta} \partial \tilde{x}^{\gamma}}$$

The second term, involving second derivatives of the coordinate transformation, is what disqualifies the Christoffel symbols from being tensor components.

Conclusion: The connection is a *rule* for differentiating tensor fields that depends on the chosen coordinate system or frame. Its transformation law reveals that it is not a tensor. However, combinations of Christoffel symbols — such as in the Riemann curvature tensor — can yield true tensorial objects.

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Conclusion: The connection is a *rule* for differentiating tensor fields that depends on the chosen coordinate system or frame. Its transformation law reveals that it is not a tensor. However, combinations of Christoffel symbols — such as in the Riemann curvature tensor — can yield true tensorial objects.

4.3.4 Torsion and Curvature

Even though the connection itself is not a tensor, we can use it to build two important objects that *are* tensors: the **torsion** and the **curvature**.

Torsion: The torsion tensor T is a $(1,2)$ tensor defined as the antisymmetric part of the connection when acting on vector fields:

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

Here, X and Y are vector fields, $\nabla_X Y$ is the covariant derivative of Y along X, and $[X,Y]$ is the Lie bracket.

In a coordinate basis $\{\partial_\mu\}$, the torsion components are:

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}$$

A connection is said to be **torsion-free** if $T^\rho_{\mu\nu} = 0$.

Curvature: The curvature tensor R is a $(1,3)$ tensor that measures the non-commutativity of covariant derivatives:

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

In a coordinate basis, the components of the Riemann curvature tensor are:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

Remarks:

- The torsion tensor vanishes for the Levi-Civita connection, which is the unique connection that is both torsion-free and metric-compatible.
- The Riemann tensor plays a central role in General Relativity, encapsulating the gravitational field in Einstein's equations.

4.3.5 The Ricci Identity

The Ricci identity describes how the covariant derivatives of tensor fields fail to commute due to curvature.

Let X^λ be a vector field. Then, the commutator of covariant derivatives acting on X^λ gives:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) X^\lambda = R^\lambda{}_{\rho\mu\nu} X^\rho$$

More generally, for a (r,s) tensor field $T^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s}$, we have:

$$[\nabla_\mu, \nabla_\nu] T^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s} = \sum_{i=1}^{r} R^{\alpha_i}{}_{\rho\mu\nu} T^{\alpha_1 \dots \rho \dots \alpha_r}_{\beta_1 \dots \beta_s} - \sum_{j=1}^{s} R^\rho{}_{\beta_j \mu\nu} T^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \rho \dots \beta_s}$$

Each index of the tensor contributes a curvature term: upper indices (contravariant) rotate with $+R$, while lower indices (covariant) rotate with $-R$.

This identity is a powerful tool in differential geometry and plays a fundamental role in deriving various geometric and physical results, including those in General Relativity.

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4.3.7 The Levi-Civita Connection

When a manifold M is equipped with a metric g, it is natural to ask whether we can define a connection that interacts nicely with the geometry that g provides.

[Fundamental Theorem of Riemannian Geometry] Let (M, g) be a smooth manifold with a metric g. There exists a unique connection ∇ such that:

1. **Metric Compatibility:** The covariant derivative of the metric vanishes:

   $$\nabla_X g = 0 \quad \text{for all vector fields } X.$$

   This ensures that the inner product between vectors is preserved under parallel transport.

2. **Torsion-Free:** The connection is symmetric in the sense that its torsion vanishes:

   $$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

This unique connection is called the **Levi-Civita connection**.

Coordinate Expression. In a local coordinate system $\{x^\mu\}$, the Levi-Civita connection is given by the Christoffel symbols:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right)$$

These symbols determine how vectors are parallel transported and how covariant derivatives act.

Example 1: Euclidean Space. Consider R^n with the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$. Then all derivatives $\partial_\rho g_{\mu\nu} = 0$, so:

$$\Gamma^\lambda_{\mu\nu} = 0$$

In this case, the Levi-Civita connection corresponds to the usual directional derivative: $\nabla_X Y = X(Y)$.

Example 2: The 2-Sphere S^2. The unit sphere in R^3 with standard coordinates (θ, ϕ) has the metric:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

The non-zero Christoffel symbols are:

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

These define how vectors change as they move along the sphere, and are essential in computing geodesics (like great circles) and curvature on S^2.

Example 3: Minkowski Spacetime. In special relativity, spacetime is modeled as R^4 with metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Here too, all metric components are constant, so the Christoffel symbols vanish and the Levi-Civita connection is flat.

These examples show how the Levi-Civita connection generalizes ordinary differentiation in flat space to curved geometries, while preserving inner products and being free of torsion.

4.3.8 The Divergence Theorem

Let M be an n-dimensional oriented Riemannian manifold with boundary ∂M, equipped with a metric g. Let X be a smooth vector field on M. Then the divergence theorem relates the integral of the divergence of X over M to a flux integral of X over the boundary ∂M:

[Divergence Theorem] Let X be a vector field on M, and let ν denote the outward-pointing unit normal to the boundary ∂M. Then

$$\int_M (\nabla \cdot X)\, \mathrm{vol}_g = \int_{\partial M} g(X, \nu)\, \mathrm{vol}_{\partial g}$$

Here:

- $\nabla \cdot X$ is the divergence of the vector field X.
- vol_g is the volume form on M induced by the metric g.
- $\mathrm{vol}_{\partial g}$ is the induced volume form on the boundary ∂M.
- $g(X, \nu)$ is the inner product of the vector field X with the unit normal ν.

Example (Euclidean R^3): In standard R^3 with the flat metric, the divergence theorem becomes the familiar statement:

$$\int_V (\nabla \cdot \vec{F})\, d^3x = \int_{\partial V} \vec{F} \cdot \hat{n}\, dS$$

where $\vec{F}$ is a vector field, $\hat{n}$ is the outward unit normal vector on the surface ∂V, and dS is the surface element.

4.3.9 The Maxwell Action

The dynamics of the electromagnetic field can be elegantly described using differential forms. Let F be the electromagnetic field strength 2-form on a Lorentzian manifold (M, g). That is,

$$F = dA$$

where A is the electromagnetic potential 1-form.

The Maxwell action is given by:

$$S = -\frac{1}{4} \int_M F \wedge \star F$$

Here:

- F is the Faraday 2-form (field strength),
- $\star F$ is the Hodge dual of F with respect to the Lorentzian metric g,
- $F \wedge \star F$ is a top-degree 4-form, which can be integrated over the 4-dimensional spacetime manifold M.

Variation and Equations of Motion. Varying the action with respect to the potential 1-form A gives:

$$\delta S = -\int_M \delta A \wedge d \star F$$

Setting this variation to zero for arbitrary δA yields Maxwell's equations in the absence of sources:

$$d \star F = 0$$

The other half of Maxwell's equations follows from the definition $F = dA$, namely:

$$dF = 0$$

which is a Bianchi identity.

This formalism highlights the geometric nature of electrodynamics and is particularly powerful in general relativistic contexts.

4.3.10 Electric and Magnetic Charges & Maxwell's Equations via Connections

In the language of differential geometry, the electromagnetic field strength is a 2-form $F \in \Lambda^2(M)$ defined on a Lorentzian manifold (M, g). It is given in terms of the gauge potential (a 1-form) A as:

$$F = dA$$

This automatically implies the **Bianchi identity** (or the absence of magnetic monopoles):

$$dF = 0$$

To incorporate electric sources, we introduce the **Hodge star** $\star$ associated with the metric g. Maxwell's equations with electric current 3-form J then read:

$$d \star F = \star J$$

This formulation implies:

- $dF = 0$ corresponds to the absence of magnetic monopoles.
- $d\star F = \star J$ gives the inhomogeneous Maxwell equations (Gauss's law and Ampère's law).

Electric and Magnetic Charges. Given a spacelike hypersurface Σ, the total electric charge Q_e enclosed in a region is:

$$Q_e = \int_\Sigma \star J = \int_{\partial\Sigma} \star F$$

by Stokes' theorem.

Similarly, if magnetic monopoles existed, a magnetic current 3-form J_m would modify the Bianchi identity:

$$dF = \star J_m \qquad \text{and} \qquad d\star F = \star J$$

Then the magnetic charge through a spatial 2-sphere S^2 would be:

$$Q_m = \int_{S^2} F$$

Maxwell's Equations via Connection on a U(1)-Bundle. More generally, A can be viewed as a connection 1-form on a principal $U(1)$-bundle over M, and the curvature of this connection is:

$$F = dA$$

This viewpoint naturally incorporates gauge invariance:

$$A \mapsto A + d\lambda \quad \Rightarrow \quad F \mapsto F$$

where λ is a scalar function (0-form). The field strength F is gauge-invariant and defines the physical observable.

This geometric formulation of electromagnetism becomes especially powerful when discussing topological effects (like Aharonov–Bohm) or generalizing to non-Abelian gauge theories.

4.3.11 Parallel Transport

Given a connection ∇ on a manifold M, we can define how vectors are "moved" along curves in a way that preserves their direction relative to the connection. This process is called **parallel transport**.

Let $\gamma : [0,1] \to M$ be a smooth curve with $\gamma(0) = p$ and $\gamma(1) = q$. A vector field $V(t)$ along the curve γ is said to be *parallel transported* if it satisfies:

$$\nabla_{\dot\gamma(t)} V(t) = 0$$

Here, $\dot\gamma(t)$ is the tangent vector to the curve at point $\gamma(t)$, and $\nabla_{\dot\gamma(t)}$ is the covariant derivative along the curve.

In coordinates, suppose $V(t) = V^\mu(t)\frac{\partial}{\partial x^\mu}$ and $\dot\gamma(t) = \frac{dx^\nu}{dt}\frac{\partial}{\partial x^\nu}$, then the parallel transport condition becomes a first-order ODE:

$$\frac{dV^\mu}{dt} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}V^\rho = 0$$

Interpretation. This equation governs how vector components change as they are moved along the curve γ, taking into account the curvature and torsion (if present) of the underlying space through the connection coefficients $\Gamma^{\mu}_{\nu\rho}$.

Properties.

- Parallel transport preserves the length and angle between vectors if the connection is compatible with a metric (i.e., $\nabla g = 0$).
- On curved manifolds, parallel transport around a closed loop generally depends on the path, a manifestation of curvature.
- This path-dependence is quantified by the Riemann curvature tensor.

4.3.12 Geodesics Revisited

A **geodesic** is a curve $\gamma(\tau)$ on a manifold M that represents the straightest possible path with respect to a given connection ∇. Formally, it satisfies the condition:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

where $\dot{\gamma} = \frac{d}{d\tau}$ is the tangent vector to the curve. This expresses that the tangent vector is parallel transported along the curve itself.

In a local coordinate system $\{x^{\mu}\}$, the geodesic equation becomes:

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} = 0$$

This is a second-order differential equation for the coordinate functions $x^{\mu}(\tau)$. A curve satisfying this equation is said to be affinely parameterised.

Angle Preservation under Parallel Transport. Let $X = \dot{\gamma}$ be the tangent vector along a geodesic $\gamma(\tau)$. If $Y(\tau)$ is a vector field that is parallel transported along the geodesic, i.e. $\nabla_X Y = 0$, then the metric compatibility of the Levi-Civita connection ($\nabla g = 0$) implies:

$$\frac{d}{d\tau}g(X,Y) = g(\nabla_X X, Y) + g(X, \nabla_X Y) = 0$$

since both $\nabla_X X = 0$ and $\nabla_X Y = 0$ along the geodesic. Therefore, the inner product $g(X,Y)$ is constant along the curve:

$$g(X(\tau), Y(\tau)) = \text{constant}$$

This result implies that the angle between Y and the tangent vector X remains unchanged as they are transported along the geodesic.

Summary. Geodesics are the curves of extremal length (or stationary action) and represent the paths that particles follow when moving under no external forces in curved spacetime. The Levi-Civita connection ensures both metric compatibility and torsion-free transport, making it the natural tool to define geodesics in Riemannian and Lorentzian geometry.

4.3.13 Normal Coordinates and the Exponential Map

Given a point $p \in M$ on a smooth manifold equipped with a connection ∇, we can construct a particularly useful coordinate system around p called **normal coordinates**. These coordinates make computations near p especially simple.

The construction is based on the **exponential map**.

Exponential Map. Let T_pM denote the tangent space at p. For each vector $v \in T_pM$, consider the unique geodesic $\gamma_v(\tau)$ satisfying:

$$\gamma_v(0) = p, \quad \dot{\gamma}_v(0) = v$$

The exponential map at p is defined as:

$$\exp_p(v) := \gamma_v(1)$$

This maps a neighborhood of the origin in T_pM diffeomorphically onto a neighborhood of p in M. The exponential map allows us to use vectors in T_pM to label points in M near p.

Normal Coordinates. Using a basis $\{e_\mu\}$ of T_pM, any vector $v \in T_pM$ can be written as $v = v^\mu e_\mu$. The coordinates of the point $\exp_p(v)$ are then defined to be $x^\mu = v^\mu$. These coordinates are called **normal coordinates** centered at p.

In these coordinates:

- $x^\mu(p) = 0$
- $g_{\mu\nu}(p) = \delta_{\mu\nu}$ (or $\eta_{\mu\nu}$ in the Lorentzian case)
- $\Gamma^\rho_{\mu\nu}(p) = 0$, i.e., the Christoffel symbols vanish at p

Interpretation. Normal coordinates make the manifold locally flat at p up to first-order derivatives. Although curvature still exists (i.e., second-order derivatives of the metric are non-zero), the vanishing of the connection coefficients at p significantly simplifies local calculations, such as Taylor expanding tensor fields or computing curvature tensors.

4.3.14 Path Dependence: Curvature and Torsion

The connection on a manifold allows us to compare vectors at different points by transporting them along curves. However, in general, this process depends on the path taken. This failure of path-independence reveals deep geometrical structures of the manifold — namely, the curvature and torsion.

Parallel Transport and Path Dependence. Let X be a vector field on a manifold M with a connection ∇. Given a vector V_p at a point $p \in M$, we can define parallel transport along a curve $\gamma : [0, 1] \to M$ using the condition:

$$\nabla_{\dot{\gamma}(t)} V(t) = 0$$

If the manifold is flat and torsion-free, the result of parallel transport depends only on the initial and final points of the curve. In general, however, parallel transporting a vector around a closed loop may lead to a different vector, indicating the presence of curvature and/or torsion.

Curvature

The curvature measures the failure of second covariant derivatives to commute. For a vector field Z and two vector fields X, Y, the Riemann curvature tensor R is defined as:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

This tensor quantifies how much the vector Z is rotated or changed when transported around an infinitesimal parallelogram spanned by X and Y.

In coordinates, the components of the Riemann tensor are:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

Torsion and Its Meaning

Torsion measures the failure of the connection to be symmetric in its lower indices. It is defined as a $(1,2)$ tensor T given by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

If $T = 0$, then the connection is said to be torsion-free. The Levi-Civita connection — the unique connection compatible with the metric and torsion-free — satisfies this condition.

In a coordinate basis, the torsion tensor components are:

$$T^\rho{}_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}$$

Interpretation of Torsion. - Curvature tells us that transporting a vector around a loop results in a rotated vector. - Torsion tells us that two paths that should close (like a parallelogram) fail to do so because the infinitesimal parallelogram does not close — its endpoints differ by the torsion vector.

Torsion can be thought of as "twisting" of the coordinate grid itself. In the presence of torsion, infinitesimal displacements do not commute:

$$[\partial_\mu, \partial_\nu] \neq 0$$

Summary.

- **Curvature** measures the failure to return to the original vector after parallel transport around a loop.
- **Torsion** measures the failure of infinitesimal parallelograms to close.
- Both are geometric obstructions to flatness: curvature is about angle change, torsion about positional offset.

4.3.15 Geodesic Deviation

The geodesic deviation equation describes how nearby geodesics on a manifold deviate from each other in the presence of curvature. It encodes tidal effects in general relativity and is governed by the Riemann curvature tensor.

Let $\gamma(\tau)$ be a geodesic with tangent vector $T = \frac{d}{d\tau} = \dot{\gamma}(\tau)$. Consider a one-parameter family of nearby geodesics, and define the separation vector field between neighboring geodesics as $J(\tau)$ — known as the **Jacobi field**.

The **geodesic deviation equation** is then:

$$\frac{D^2 J^\mu}{D\tau^2} + R^\mu{}_{\nu\rho\sigma} T^\nu J^\rho T^\sigma = 0$$

or more compactly using covariant derivatives:

$$\nabla_T \nabla_T J + R(T, J)T = 0$$

Interpretation. - J^μ describes the infinitesimal separation between neighboring geodesics. - The second covariant derivative $\frac{D^2 J^\mu}{D\tau^2}$ measures the relative acceleration between geodesics. - The curvature term $R^\mu{}_{\nu\rho\sigma} T^\nu J^\rho T^\sigma$ encodes how the curvature of spacetime bends geodesics toward or away from each other.

Physical Meaning. In general relativity, geodesic deviation corresponds to the presence of tidal forces. Two nearby freely falling particles in a gravitational field will accelerate relative to one another due to spacetime curvature. This is how curvature manifests in physical measurements.

4.3.16 More on the Riemann Tensor and its Friends

When we lower an index on the Riemann tensor using the metric $g_{\mu\nu}$, we define the **fully covariant Riemann tensor** as:

$$R_{\sigma\rho\mu\nu} = g_{\sigma\lambda} R^\lambda{}_{\rho\mu\nu}$$

This object possesses several important symmetries:

- **Antisymmetry in the last two indices:**

$$R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu}$$

- **Antisymmetry in the first two indices:**

$$R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu}$$

- **Symmetry under exchange of index pairs:**

$$R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$$

- **First Bianchi identity (cyclic identity):**

$$R_{\sigma[\rho\mu\nu]} = 0$$

  which expands as

$$R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0$$

Second Bianchi Identity. The Riemann tensor satisfies a differential identity known as the **second Bianchi identity**:

$$\nabla_{[\lambda}R_{\sigma\rho]\mu\nu} = 0$$

Alternatively, using antisymmetrisation on the last indices:

$$R_{\sigma\rho[\mu\nu;\lambda]} = 0$$

These identities play a central role in the structure of general relativity. In particular, they are instrumental in deriving the Einstein field equations from the Einstein–Hilbert action.

4.4 Connection 1-Forms and Curvature 2-Forms

4.4.1 Vielbeins

On a smooth n-dimensional manifold M equipped with a metric g, we can introduce an orthonormal frame of 1-forms $\{e^a\}$, known as **vielbeins** (or **frame fields**), such that the metric takes the form

$$g = \eta_{ab}\, e^a \otimes e^b$$

where η_{ab} is the Minkowski (or Euclidean) metric depending on the signature of spacetime. The vielbeins relate the coordinate basis dx^μ to an orthonormal basis via:

$$e^a = e^a{}_\mu\, dx^\mu \quad \text{and its inverse} \quad dx^\mu = e^\mu{}_a\, e^a$$

where $e^a{}_\mu$ and $e^\mu{}_a$ are inverse matrices.

4.4.2 Connection 1-Forms

To define parallel transport and curvature in the vielbein formalism, we introduce the **connection 1-forms** $\omega^a{}_b$, which are Lie-algebra-valued 1-forms encoding how frames twist and rotate over the manifold. They satisfy the structure equation:

$$de^a + \omega^a{}_b \wedge e^b = T^a$$

where T^a is the **torsion 2-form**. For the torsion-free (Levi-Civita) connection, this reduces to the first Cartan structure equation:

$$de^a + \omega^a{}_b \wedge e^b = 0$$

The connection 1-forms satisfy the antisymmetry property:

$$\omega_{ab} = -\omega_{ba} \quad \text{with} \quad \omega_{ab} := \eta_{ac}\omega^c{}_b$$

4.4.3 Curvature 2-Forms

The curvature is captured by the **curvature 2-forms** $\Omega^a{}_b$, defined via the second Cartan structure equation:

$$\Omega^a{}_b := d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

These forms encode the Riemann curvature tensor in the frame basis. Specifically, the Riemann tensor components can be recovered from:

$$\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d$$

Thus, the curvature 2-form expresses the infinitesimal holonomy: how much a vector rotates after being parallely transported around an infinitesimal parallelogram.

4.4.4 Summary of Cartan's Structure Equations

- **First structure equation (torsion):**

$$T^a = de^a + \omega^a{}_b \wedge e^b$$

- **Second structure equation (curvature):**

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

These equations form the foundation of the geometric interpretation of gravity in the vielbein (tetrad) formalism, and are central to formulations like Einstein-Cartan theory and gauge theories of gravity.

4.5 An Example: The Schwarzschild Metric

One of the most important exact solutions of Einstein's equations is the Schwarzschild metric. It describes the spacetime geometry outside a static, spherically symmetric, non-rotating mass.

In Schwarzschild coordinates (t, r, θ, ϕ), the line element is given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Here: - G is the gravitational constant, - M is the mass of the central object, - r is the radial coordinate, - θ and ϕ are the usual angular coordinates on the sphere.

Key Features

- The coordinate singularity at $r = 2GM$ is the **Schwarzschild radius**, often denoted r_s. This defines the event horizon of a black hole.
- The curvature singularity is at $r = 0$, where tidal forces become infinite.
- In the limit $r \to \infty$, the spacetime becomes asymptotically flat:

$$ds^2 \to -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

- The metric is static and spherically symmetric, meaning it has timelike Killing vector ∂_t and rotational symmetry.
- The non-zero Christoffel symbols, Riemann tensor, Ricci tensor, and Einstein tensor for this metric encode the vacuum nature ($R_{\mu\nu} = 0$) of the spacetime outside the mass.

Interpretation

This solution solves the vacuum Einstein equations $R_{\mu\nu} = 0$, meaning it describes the geometry of spacetime in a region where no matter is present, but a central mass curves spacetime.

Important physical consequences:

- Precession of planetary orbits (e.g., Mercury),
- Gravitational time dilation,
- Light bending near massive objects,
- Black hole physics, including event horizons and Hawking radiation (semi-classically).

4.6 The Relation to Yang–Mills Theory

There is a striking formal similarity between general relativity and Yang–Mills theory when viewed through the lens of differential geometry and gauge theory.

In general relativity, the geometry of spacetime is encoded in the spin connection $\omega^a{}_b$, and curvature is described by the curvature 2-form:

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

This is structurally analogous to the field strength F in Yang–Mills theory:

$$F = dA + A \wedge A$$

where A is the Lie-algebra-valued connection (gauge potential) for some internal symmetry group.

Comparison Table

	General Relativity	**Yang–Mills Theory**
Gauge Group	$\mathrm{SO}(n)$ or $\mathrm{SO}(1, n-1)$	Compact Lie group (e.g., $\mathrm{SU}(N)$)
Connection	Spin connection $\omega^a{}_b$	Gauge field A^a
Curvature	$\Omega^a{}_b$	Field strength F^a
Bianchi Identity	$D\Omega^a{}_b = 0$	$DF^a = 0$
Action	Einstein–Hilbert: $\int R \star 1$	Yang–Mills: $\int \mathrm{Tr}(F \wedge \star F)$

This analogy reveals gravity as a kind of gauge theory, though with important differences: the "gauge group" in gravity acts on the tangent bundle (spacetime itself), while in Yang–Mills it acts on an internal vector space.

Remarks

- In gravity, the vielbein e^a plays a dual role: it encodes both the metric and allows coupling to spinors.
- In contrast to Yang–Mills theory, the spin connection in gravity is not an independent field in the metric formalism—it is determined by e^a via the torsion-free condition.

5 Einstein's Equations

It is now time to do some physics. The force of gravity is mediated by a gravitational field. The glory of general relativity is that this field is identified with a metric $g_{\mu\nu}(x)$ on a four-dimensional Lorentzian manifold that we call *spacetime*.

This metric is not fixed; it is, like all other fields in nature, a *dynamical object*. This means that there are rules which govern how this field evolves in time. The purpose of this section is to explore these rules and some of their consequences.

We will start by understanding the dynamics of the gravitational field in the absence of any matter. We will then turn to understand how the gravitational field responds to matter — or, more precisely, to energy and momentum — in Section 4.5.

5.1 The Einstein-Hilbert Action

All our fundamental theories of physics are described by action principles. Gravity is no different. Furthermore, the straight-jacket of differential geometry places enormous restrictions on the kind of action that we can write down. These restrictions ensure that the action is something intrinsic to the metric itself, rather than depending on any choice of coordinates.

Spacetime is a manifold M, equipped with a metric of Lorentzian signature. An action is an integral over M. We know from Section 2.4.4 that we need a volume form to integrate over a manifold. Happily, as we have seen, the metric provides a canonical volume form, which we can multiply by any scalar quantity. Given that we only have the metric to play with, the simplest such (non-trivial) quantity is the Ricci scalar R. This motivates the wonderfully concise action

$$S = \int_M d^4x \sqrt{-g}\, R \,, \tag{19}$$

known as the *Einstein–Hilbert action.* Note that the minus sign under the square root appears because we are working in a Lorentzian spacetime, where the metric has a single negative eigenvalue, making its determinant $g = \det g_{\mu\nu}$ negative.

As a quick sanity check, recall that the Ricci tensor takes the schematic form

$$R \sim \partial\Gamma + \Gamma\Gamma \,,$$

while the Levi-Civita connection itself is

$$\Gamma \sim \partial g \,.$$

This means that the Einstein–Hilbert action is second order in derivatives, just like most other fundamental action principles in physics.

5.2 Varying the Einstein–Hilbert Action

We now vary the Einstein–Hilbert action (19) to obtain the Euler–Lagrange equations.

We consider an infinitesimal shift in the metric:

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \,.$$

Writing the Ricci scalar as $R = g^{\mu\nu}R_{\mu\nu}$, the variation of the action is

$$\delta S = \int_M d^4x \Big[\underbrace{\delta(\sqrt{-g})g^{\mu\nu}R_{\mu\nu}}_{\text{(a)}} + \underbrace{\sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu}}_{\text{(b)}} + \underbrace{\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}}_{\text{(c)}} \Big] .$$

It is convenient to work with variations of the *inverse metric.* The relation between $\delta g_{\mu\nu}$ and $\delta g^{\mu\nu}$ is

$$g_{\rho\mu}g^{\mu\nu} = \delta_\rho^{\ \nu} \Rightarrow \delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} .$$

We now compute the first term.

Claim. The variation of $\sqrt{-g}$ is

$$\delta\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} .$$

This allows us to rewrite the first term as

$$\delta(\sqrt{-g})g^{\mu\nu}R_{\mu\nu} = -\tfrac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\mu\nu}R_{\mu\nu}\,\delta g^{\mu\nu} = -\tfrac{1}{2}\sqrt{-g}R\,g_{\mu\nu}\delta g^{\mu\nu} .$$

The second term is already in the desired form:

$$\sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} = \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} .$$

The final term, $\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$, involves the variation of the Ricci tensor and can be expressed as a total derivative:

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{-g}\nabla_\rho\Big(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\nu_{\mu\nu}\Big).$$

This term gives only a boundary contribution, which we can discard if we assume $\delta g^{\mu\nu} = 0$ on the boundary of M.

Putting this together, the total variation is

$$\delta S = \int_M d^4x\,\sqrt{-g}\Big[R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}\Big]\delta g^{\mu\nu}.$$

Requiring $\delta S = 0$ for arbitrary $\delta g^{\mu\nu}$ gives the vacuum Einstein field equations:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 0 .$$

Claim. The variation of the Ricci tensor is a total derivative:

$$\delta R_{\mu\nu} = \nabla_\rho\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho_{\mu\rho} ,$$

with

$$\delta\Gamma^\rho_{\mu\nu} = \tfrac{1}{2}g^{\rho\sigma}\Big(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\nu}\Big).$$

This confirms that the term $\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$ is a total derivative, yielding only a boundary contribution upon integration.

The upshot of these calculations is that

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu X^\mu \qquad \text{with } X^\mu = g^{\rho\nu}\delta\Gamma^\mu_{\rho\nu} - g^{\mu\nu}\delta\Gamma^\rho_{\nu\rho}.$$

Then, the variation of the action (19) can be expressed as

$$\delta S = \int_M d^4x\,\sqrt{-g}\Big[(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu})\,\delta g^{\mu\nu} + \nabla_\mu X^\mu\Big].$$

The final term is a total derivative and, by the divergence theorem (see Section 3.2.4), can be ignored when $\delta g^{\mu\nu} = 0$ on the boundary. Hence, the Euler–Lagrange equations are

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 0\,,$$

with $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ the *Einstein tensor.* These are the Einstein field equations in vacuum.

Taking the trace of this expression gives $R = 0$, implying that the vacuum equations reduce to

$$R_{\mu\nu} = 0\,,$$

i.e., the metric is *Ricci–flat.*

An Aside on Dimensional Analysis

In units where $c = \hbar = 1$, the metric $g_{\mu\nu}$ is dimensionless, and the Ricci scalar R has units of [length^{-2}]. The Einstein–Hilbert action

$$S = \int d^4x\,\sqrt{-g}R$$

is then of order [length^2]. To match the units of a physical action, one introduces Newton's constant G, yielding

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R\,,$$

with G having units [length^2]. In this form, the total action is dimensionless.

The Cosmological Constant

One can generalize the Einstein–Hilbert action by adding a constant term:

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}(R - 2\Lambda)\,,$$

where Λ is the *cosmological constant.* The resulting field equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\,T_{\mu\nu}\,,$$

with $\Lambda g_{\mu\nu}$ acting like a vacuum energy density.

Higher–Derivative Terms

The Einstein–Hilbert action contains terms with at most second derivatives of the metric. In more general theories of gravity, one can consider higher–order corrections, such as

$$\mathcal{L}_{\text{high-deriv.}} = \sqrt{-g}\Big[R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \dots\Big]\,.$$

These lead to fourth–order field equations and arise in effective theories of gravity and quantum corrections.

Diffeomorphisms Revisited

The action of General Relativity is diffeomorphism invariant: for any smooth coordinate map $x^\mu \to x'^\mu(x)$, the form of the action and the field equations remain unchanged. This is the formal statement that General Relativity is a *covariant theory*, making the metric a truly geometric, coordinate–independent object.

The Einstein Equations with Matter

So far, we have considered the dynamics of the metric $g_{\mu\nu}$ in the absence of matter, yielding the vacuum Einstein equations:

$$G_{\mu\nu} := R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 0\,.$$

To describe how matter influences the geometry of spacetime, we add a matter action

$$S_{\text{matter}}[g, \psi, A, \dots] = \int d^4x\,\sqrt{-g}\,\mathcal{L}_{\text{matter}}(g, \psi, A, \dots)\,,$$

where $\psi, A, \dots$ denote generic matter fields (scalars, spinors, gauge fields, etc.).

Varying the matter action with respect to the metric defines the *energy–momentum tensor*:

$$T_{\mu\nu} := -\tfrac{2}{\sqrt{-g}} \tfrac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}\,.$$

Combining the matter and gravity variations, the total action

$$S_{\text{tot}} = \tfrac{1}{16\pi G} \int d^4x\,\sqrt{-g} R + S_{\text{matter}}$$

yields the *Einstein field equations with matter*:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,.$$

These are the fundamental equations of general relativity, capturing the interplay between matter, energy, and the geometry of spacetime.

Examples of the Energy–Momentum Tensor

The energy–momentum tensor $T_{\mu\nu}$ encodes the density and flux of energy and momentum for any matter present. Its form depends on the matter fields in question:

- **Scalar Field:** For a scalar field ϕ,

$$T_{\mu\nu} = \nabla_\mu \phi\, \nabla_\nu \phi - g_{\mu\nu} \Big[\tfrac{1}{2} g^{\rho\sigma} \nabla_\rho \phi\, \nabla_\sigma \phi + V(\phi)\Big].$$

- **Electromagnetic Field:** For the Maxwell field $F_{\mu\nu}$,

$$T_{\mu\nu} = \tfrac{1}{4\pi}\Big(F_{\mu\rho} F_\nu{}^\rho - \tfrac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\Big).$$

- **Perfect Fluid:** For a fluid with rest–frame energy density ρ, pressure p, and four–velocity u^μ,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\, g_{\mu\nu}.$$

More generally, one can consider any matter content as arising from a suitable matter action yielding an associated $T_{\mu\nu}$.

The Slippery Business of Energy Conservation

In general relativity, energy conservation is a more nuanced concept than in Newtonian mechanics. In special relativity, we have

$$\partial_\mu T^{\mu\nu} = 0\,,$$

expressing the local conservation of energy and momentum. However, in curved spacetime this generalizes to

$$\nabla_\mu T^{\mu\nu} = 0\,,$$

which captures the fact that matter and energy evolve covariantly with the geometry of spacetime.

This covariant conservation law is built into the Einstein field equations via the Bianchi identities:

$$\nabla_\mu G^{\mu\nu} = 0 \Rightarrow \nabla_\mu T^{\mu\nu} = 0\,.$$

Yet, this is not a global conservation statement: in a curved, dynamic spacetime, one generally *cannot* define a global, coordinate–independent total energy for the gravitational field itself. The energy of matter is well–defined at each point via $T^{\mu\nu}$, but the energy of gravity is intrinsically tied to the geometry and does not have a localized, tensorial description.

This is one of the profound conceptual shifts introduced by general relativity — energy and momentum are still conserved, but only in the covariant, local sense. Globally, in an expanding universe or highly dynamical spacetime, the traditional notions of energy conservation must be treated with care.

6 The Schwarzschild Solution

The simplest and most important solution of the vacuum Einstein equations,

$$R_{\mu\nu} = 0\,,$$

is the *Schwarzschild solution*. It describes the spacetime geometry surrounding a static, spherically symmetric, uncharged massive object.

Assuming spherical symmetry and staticity, one can write the metric ansatz as

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,.$$

In vacuum ($R_{\mu\nu} = 0$), one finds

$$e^{2\Phi(r)} = e^{-2\Lambda(r)} = 1 - \frac{2GM}{r}\,,$$

and the Schwarzschild metric becomes

$$ds^2 = -\Big(1 - \tfrac{2GM}{r}\Big)dt^2 + \Big(1 - \tfrac{2GM}{r}\Big)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,.$$

Interpretation

Here, M is the mass of the central object. The surface $r = 2GM$ is called the Schwarzschild *event horizon*. It marks the point of no return for any infalling matter or light. As $r \to \infty$, the metric approaches the flat spacetime of special relativity, making M the total (ADM) mass of the spacetime.

Physical Significance

The Schwarzschild solution was the first exact solution of the Einstein field equations and is a cornerstone of general relativity. It describes:

- The spacetime around stars and planets (for $r \gg 2GM$).
- The geometry of black holes ($r \leq 2GM$).
- The bending of light and the precession of planetary orbits.

With this solution, we gain the foundation for understanding black holes, gravitational lensing, and relativistic orbits — making it one of the central results of general relativity.

Komar Mass of the Schwarzschild Black Hole

The Schwarzschild metric,

$$ds^2 = -\left(1 - \tfrac{2GM}{r}\right)dt^2 + \left(1 - \tfrac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,,$$

reduces for large r to the Newtonian potential $\Phi(r) = -\frac{GM}{r}$, confirming its interpretation as the gravitational field of a point mass M at the origin.

We can compute the total mass M of the Schwarzschild spacetime using the *Komar integral*. This approach exploits the timelike Killing vector $K = \partial_t$, with its associated one–form

$$K_\mu dx^\mu = g_{00} dt = -\left(1 - \tfrac{2GM}{r}\right)dt\,.$$

From this, we construct the two–form

$$F = dK = -\tfrac{2GM}{r^2} dr \wedge dt\,,$$

which has precisely the $1/r^2$ behavior reminiscent of an electric field. According to the Komar construction, the total mass enclosed within any sphere S^2 of radius $r > 2GM$ is

$$M_{\text{Komar}} = -\tfrac{1}{8\pi G} \int_{S^2} \star dK\,.$$

Evaluating the integral gives

$$M_{\text{Komar}} = M\,,$$

and this result is independent of the sphere's radius, just as the total charge in Maxwell theory is independent of the surface used for its measurement.

Interestingly, this formalism mimics Maxwell's theory, yielding an effective "charge" M_{Komar}. Since $d \star F = 0$ is satisfied in vacuum, one might expect $M_{\text{Komar}} = 0$. However, the "charge" is sourced by the singularity at $r = 0$, making the Schwarzschild solution akin to a point charge located at the origin.

Although the metric is a valid solution of the Einstein equations for any M, only $M \geq 0$ gives a physically sensible spacetime. The $M = 0$ case is just flat Minkowski space, and the $M < 0$ solution contains a naked singularity with pathologies that make it physically unacceptable.

Birkhoff's Theorem

An important result in General Relativity is *Birkhoff's Theorem*. It states that any spherically symmetric solution of the vacuum Einstein equations,

$$R_{\mu\nu} = 0\,,$$

must be static and asymptotically flat, and is therefore uniquely described by the Schwarzschild metric.

More generally:

- Even if the matter distribution is spherically symmetric but dynamical (such as a pulsating star), the exterior spacetime remains Schwarzschild.
- In other words, a spherically symmetric vacuum solution has no *monopole gravitational radiation*. This means that changes within the spherically symmetric matter do not affect the exterior metric, as long as the matter remains spherically symmetric.

Birkhoff's Theorem guarantees that Schwarzschild is the unique vacuum solution for any spherically symmetric configuration, making it a cornerstone for understanding black holes and their dynamics.

Coordinate Singularity vs. True Singularity

The Schwarzschild metric appears to "go bad" at two special radii:

$$r = 2GM \qquad \text{and} \qquad r = 0\,.$$

At $r = 2GM$, the metric component $g_{tt} \to 0$ and $g_{rr} \to \infty$. However, this is merely a *coordinate singularity*: it can be removed by choosing a different coordinate chart (e.g., Kruskal–Szekeres coordinates). In this sense, $r = 2GM$ is a regular surface — the *event horizon* of the black hole.

In contrast, at $r = 0$, curvature invariants such as the Kretschmann scalar

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty$$

diverge, indicating a genuine *physical singularity*. Here the theory of general relativity itself breaks down. To make sense of physics in this regime, one must ultimately turn to a quantum theory of spacetime.

The Near Horizon Limit: Rindler Space

Near the Schwarzschild horizon $r = 2GM$, the geometry of spacetime simplifies dramatically. Let us set

$$r = 2GM + \rho\,,$$

with $\rho \ll 2GM$. The Schwarzschild metric then becomes

$$ds^2 = -\left(1 - \tfrac{2GM}{2GM+\rho}\right)dt^2 + \left(1 - \tfrac{2GM}{2GM+\rho}\right)^{-1}dr^2 + (2GM+\rho)^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

To leading order in ρ,

$$ds^2 \approx -\tfrac{\rho}{2GM}dt^2 + \tfrac{2GM}{\rho}d\rho^2 + (2GM)^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Introduce the proper radial coordinate

$$\chi = 2\sqrt{GM\rho}\,,$$

such that

$$ds^2 \approx -\tfrac{\chi^2}{16G^2M^2}dt^2 + d\chi^2 + (2GM)^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

The $t - \chi$ part of the metric is precisely the Rindler metric,

$$ds^2_{\text{Rindler}} = -\kappa^2\chi^2 dt^2 + d\chi^2\,,$$

with surface gravity

$$\kappa = \tfrac{1}{4GM}\,.$$

This shows that near the horizon, the Schwarzschild black hole is well–approximated by a patch of Rindler spacetime, capturing the physics of a uniformly accelerated observer. In this regime, many results, including the Unruh effect and the thermal nature of black hole horizons, find a natural and geometrically elegant description.

7 Simulation of Schwarzschild Orbits

To complement the theoretical discussion of general relativity, we developed simulations of particle and photon orbits around a Schwarzschild black hole. These simulations visualize how the curvature of spacetime governs the geodesic motion of objects.

2D Orbital Dynamics in Schwarzschild Geometry

To simulate particle orbits around a Schwarzschild black hole in two dimensions, we begin by analyzing the effective potential and deriving the corresponding orbital equations.

Effective Potential

The radial motion of a test particle in Schwarzschild spacetime can be described by the conservation of energy:

$$\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) = E^2$$

where E is the conserved specific energy of the particle, and $V_{\text{eff}}(r)$ is the effective potential, given by:

$$V_{\text{eff}}(r) = \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right)$$

Here, M is the mass of the black hole, L is the conserved angular momentum, and $\epsilon = 1$ for a massive particle and $\epsilon = 0$ for a photon.

Derivative of the Effective Potential

To find the radial acceleration, we differentiate the effective potential with respect to r:

$$\frac{dV_{\text{eff}}}{dr} = \frac{2M}{r^2}\left(\epsilon + \frac{L^2}{r^2}\right) - \left(1 - \frac{2M}{r}\right)\cdot\frac{2L^2}{r^3}$$

Conversion to Angular Equation

To express the motion in terms of the angular coordinate ϕ, we use conservation of angular momentum:

$$\frac{d\phi}{d\tau} = \frac{L}{r^2} \quad \Rightarrow \quad \frac{dr}{d\tau} = \frac{L}{r^2} \cdot \frac{dr}{d\phi}$$

Squaring both sides:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{L}{r^2}\right)^2 \left(\frac{dr}{d\phi}\right)^2$$

Substituting into the energy equation:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2} \left(E^2 - V_{\text{eff}}(r)\right)$$

Differentiating with respect to ϕ:

$$2\frac{dr}{d\phi} \cdot \frac{d^2r}{d\phi^2} = \frac{d}{d\phi} \left[\frac{r^4}{L^2} \left(E^2 - V_{\text{eff}}(r)\right)\right]$$

$$= \frac{1}{L^2} \left[4r^3 \frac{dr}{d\phi}(E^2 - V_{\text{eff}}) - r^4 \frac{dV_{\text{eff}}}{dr}\frac{dr}{d\phi}\right]$$

Canceling $\frac{dr}{d\phi}$:

$$\frac{d^2r}{d\phi^2} = \frac{1}{2L^2} \left[4r^3(E^2 - V_{\text{eff}}) - r^4 \frac{dV_{\text{eff}}}{dr}\right]$$

In the case of circular orbits or near turning points where $E^2 \approx V_{\text{eff}}$, the first term vanishes:

$$\boxed{\frac{d^2r}{d\phi^2} = -\frac{1}{2} \cdot \frac{dV_{\text{eff}}}{dr} \cdot \frac{r^4}{L^2}}$$

Radial Velocity

We can also compute the radial velocity from the energy equation:

$$\frac{dr}{d\tau} = \sqrt{E^2 - V_{\text{eff}}(r)}$$

This helps us determine whether the particle is moving inward or outward at a given point in the orbit.

Significance of the Equation

The final angular equation allows us to numerically integrate the orbital path of a particle around a Schwarzschild black hole, using the second-order ODE in terms of ϕ. This approach is particularly useful for visualizing precessing orbits and photon trajectories in a 2D plane.

Sample Simulation Output

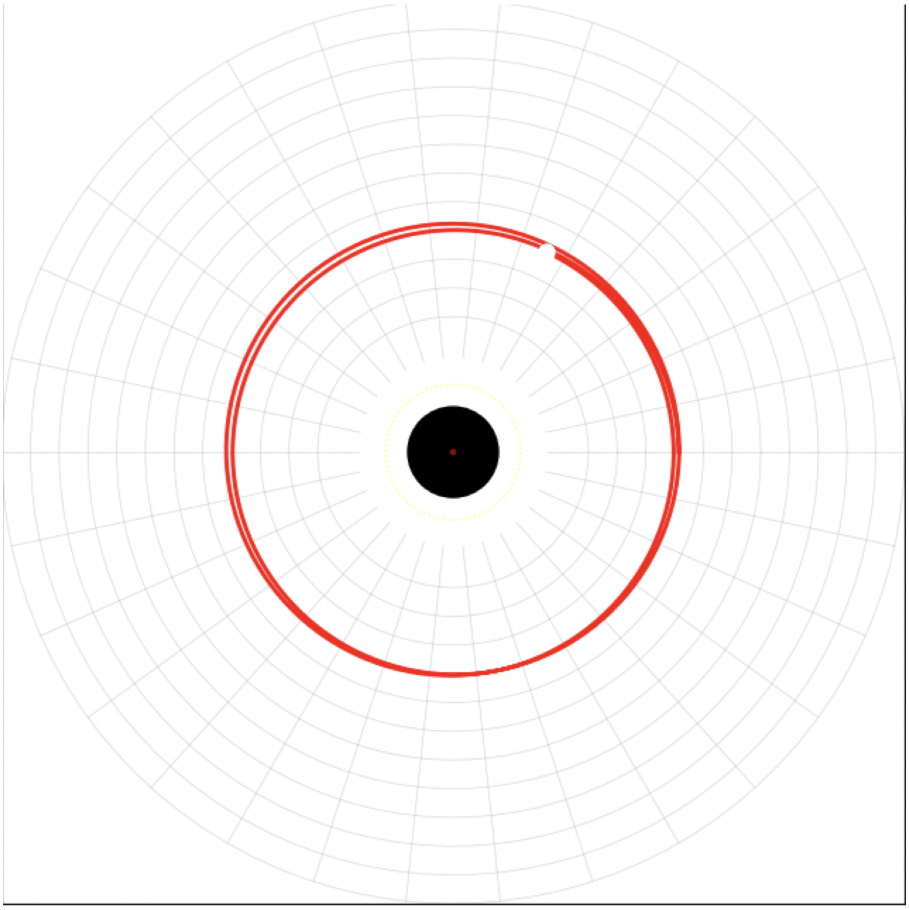


Figure 3: Trajectory of a massive particle orbiting a Schwarzschild black hole.

Parameters Used:

- $M = 0.5$ *(Mass of black hole)*
- $L = 1.8867$ *(Angular momentum)*
- $r_0 = 5$ *(Initial radius)*
- $e = 1$ *(Massive particle)*
- $E = 0.9560$ *(Energy)*
- inward = True *(Initial direction of radial velocity)*

7.1 3D Visualisation Using `Manim`

To better capture the curvature of spacetime, we also implemented a 3D animated version of the orbits using the `manim` engine. This version presents a rotating spatial view of the orbit, rendered over a Schwarzschild curvature surface for aesthetic clarity.

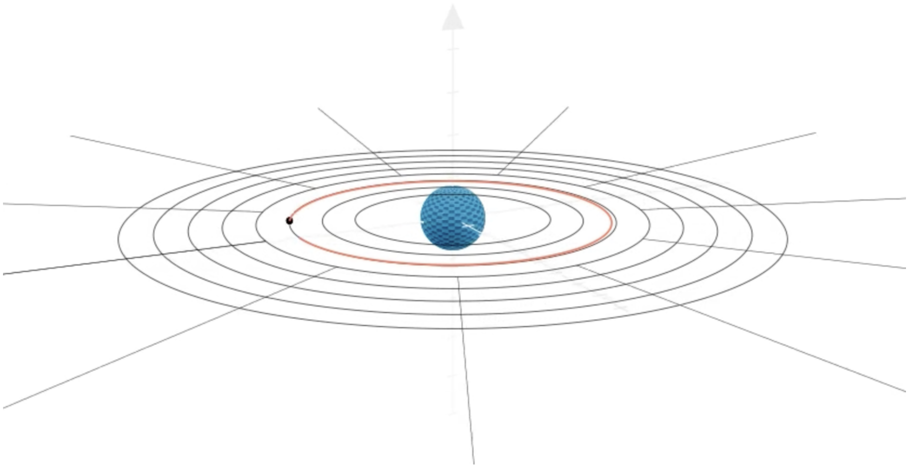
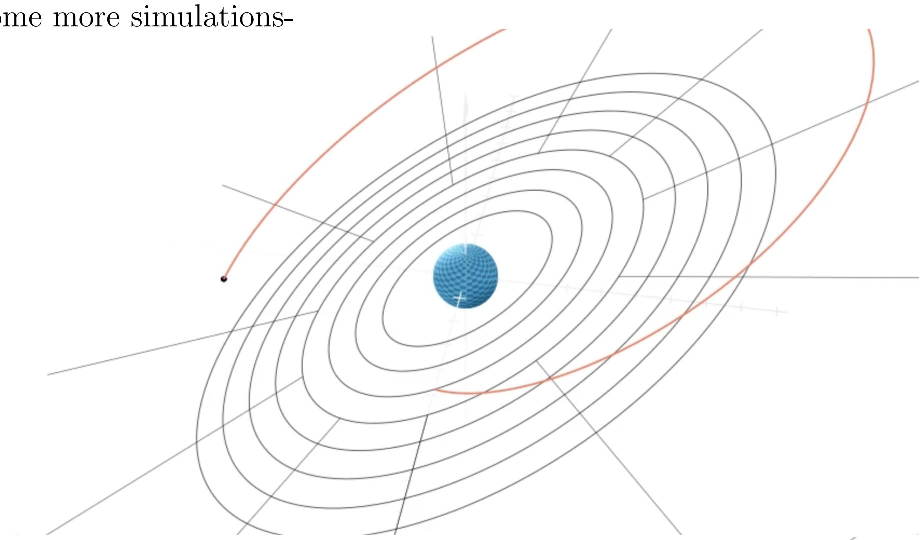
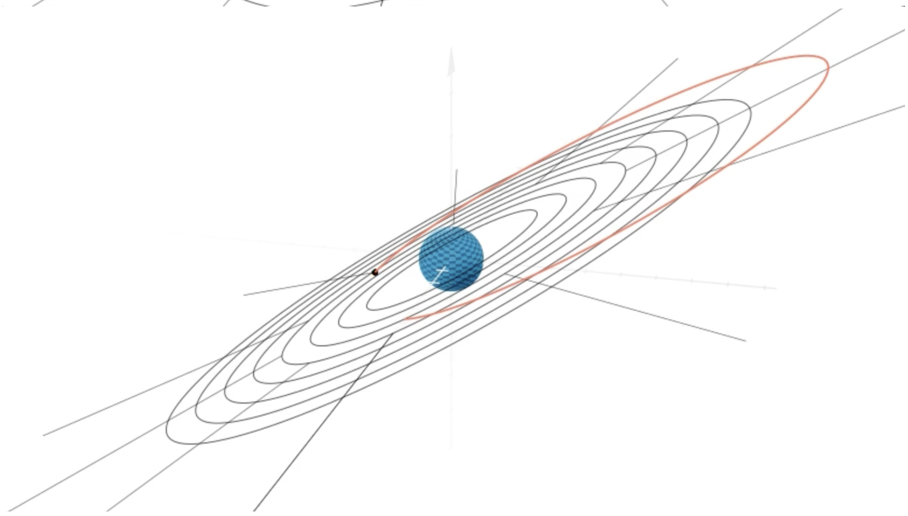


figure3D rendered orbit around a Schwarzschild black hole using Manim with the same paramaters as the previous ones.

Some more simulations-





7.2 Photon vs Massive Particle Geodesics

Our code includes a toggle to simulate either photons ($\epsilon = 0$) or massive particles ($\epsilon = 1$), affecting the structure of the effective potential:

$$V_{\text{eff}}(r) = \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right)$$

The orbits reflect general relativistic effects such as:

- Light bending
- Photon sphere at $r = 3M$
- Perihelion precession for near-circular orbits (e.g., Mercury)

7.3 Code Repository

All simulation code and instructions are available at: `[Google Drive Link]`

It contains:

Code and Simulation Files

The following Python files are used for visualizing the Schwarzschild orbits:

- `visualisation_2d.py` — Matplotlib version for 2D trajectory simulation.
- `visualisation_3d.py` — Manim 3D scene for animated rendering of orbits.
- Sample rendered videos and figures are also included.

How to Run the Simulations:

- For the 2D version:
  `python visualisation_2d.py`
- For the 3D version using Manim:
  `manim -pql visualisation_3d.py SchwarzschildOrbit`