Rotating stellar structure

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1 The Stellar Equation for a Uniformly Rotating Star

The force balance equation for each volume element for a star rotating with angular velocity ω, is:

$$-\nabla P + \rho \mathbf{g} = \rho \ \omega \times (\omega \times \mathbf{r}) \tag{1}$$

where P is the pressure , ρ is the density, $\mathbf{g}$ is the gravitational field, and $\mathbf{r}$ is the position in space for that volume element. Using:

$$\nabla \cdot \mathbf{g} = -4\pi G\rho, \quad \nabla \times \mathbf{g} = 0 \tag{2}$$

and the equation of state $\rho = \rho(P)$ we can have a double differential equation that is computationally feasible under right boundary conditions:

$$\nabla^2 P - \frac{\rho'}{\rho}(\nabla P \cdot \nabla P) + 4\pi G\rho^2 - 2\omega^2\rho = 0 \tag{3}$$

The pressure near the center and its derivatives can be used as boundary condition:

$$P(dr, \theta, \phi) = P_c - \frac{2\pi G}{3}\rho(P_c)^2 dr^2 + \frac{1}{2}\rho(P_c)\omega^2 dr^2 \sin^2\theta \tag{4}$$

2 The Computational Method

Introducing dimensionless variables:

$$p = \frac{P}{P_0}, \quad h = \frac{\rho}{\rho_0}, \quad \xi = r\sqrt{\frac{4\pi G\rho_0^2}{P_0}}, \quad w = \frac{\omega}{\sqrt{4\pi G\rho_0}} \tag{5}$$

the structure equation becomes:

$$\nabla^2 p - \frac{h'}{h}(\nabla p \cdot \nabla p) + h^2 - 2w^2 h = 0 \tag{6}$$

Assuming azimuthal symmetry, the structure equation in spherical coordinates is:

$$\frac{\partial^2 p}{\partial \xi^2} + \frac{1}{\xi^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{2}{\xi}\frac{\partial p}{\partial \xi} + \frac{1}{\xi^2 \tan\theta}\frac{\partial p}{\partial \theta} - \frac{h'}{h}\left(\frac{\partial p}{\partial \xi}\right)^2 - \frac{h'}{h\xi^2}\left(\frac{\partial p}{\partial \theta}\right)^2 + h^2 - 2w^2 h = 0 \tag{7}$$

Substituting $\mu = \cos\theta$ into the equation:

$$\frac{\partial^2 p}{\partial \xi^2} + \frac{1-\mu^2}{\xi^2}\frac{\partial^2 p}{\partial \mu^2} + \frac{2}{\xi}\frac{\partial p}{\partial \xi} + \frac{2\mu}{\xi^2}\frac{\partial p}{\partial \mu} - \frac{h'}{h}\left(\frac{\partial p}{\partial \xi}\right)^2 - \frac{h'}{h}\frac{1-\mu^2}{\xi}\left(\frac{\partial p}{\partial \mu}\right)^2 + h^2 - 2w^2 h = 0 \tag{8}$$

Solving iteratively from the center outward, using initial conditions:

$$p(\xi = 0, \mu) = p_c \tag{9}$$

$$\left(\frac{1}{\xi}\frac{\partial p}{\partial \xi}\right)_{\xi=0} = -\frac{1}{3}(h(p_c))^2 + h(p_c)w^2(1-\mu^2) \tag{10}$$

$$\left(\frac{1}{\xi^2}\frac{\partial p}{\partial \mu}\right)_{\xi=0} = h(p_c)w^2\mu \tag{11}$$

$$\left(\frac{1}{\xi^2}\frac{\partial^2 p}{\partial \mu^2}\right)_{\xi=0} = h(p_c)w^2 \tag{12}$$

The star is solved computationally from the differential equation and the binary condition we have obtained layer by layer using the finite difference method. The pressure at the first few discrete steps of ξ are obtained directly using the boundary condition. Now, if the pressure values at a given layer (ξ) is known them the values of $\frac{\partial p}{\partial \mu}$, $\frac{\partial^2 p}{\partial \mu^2}and\frac{\partial p}{\partial \xi}$ can be calculated. These values are then fed into the differential equation to obtain $\frac{\partial^2 p}{\partial \xi^2}$. Using this, the values of p as well as that of $\frac{\partial p}{\partial \xi}$ can be updated. This way, the next layer of the star could be obtained.

The biggest hurdle we faced whilst using this method is that any small computational inaccuracies are amplified over the course of iterations, forming discontinuities or outliers in our data at certain values of μ. These outliers in turn give absurdly high values of $\frac{\partial p}{\partial \mu}and\frac{\partial^2 p}{\partial \mu^2}$, and the pressure profile is found to not converge to 0 for higher values of w. This problem was partially overcome by modifications such as using less values of ξ, using a larger step size to decrease the number of iterations, smoothening the function by fitting a polynomial onto it, implementing the Range-Kutta Method over Euler's method, etc. Hence, for a good workable range of p_c and w, we were able to solve the star.

As for the equation of state, most of our calculations was concerning with non-relativistic white dwarf (mass much lower than the Chandrasekhar Mass Limit). In these stars, the pressure is constituted by the degeneracy pressure of non-relativistic ideal gas of electrons. The equation of state can be theoretically calculated to be of the form $P \propto \rho^{\frac{5}{3}}$, with the constant of proportionality being a universal constant[1] We also applied this method on neutron star, where the density is assumed to be constant throughout.

3 Chandrasekhar mass limit for rotating white dwarfs

We need to first start by looking at the non rotating case of the white dwarf and computationally obtain the Chandrasekhar mass limit in this case and then apply this computational method to the rotating star.

3.1 The non-rotating case

Here we consider a typical white dwarf where the gravitational pull of the compact star is balanced by the degeneracy pressure from the Fermi gas of electrons. The pressure of the gas is given by:

$$\frac{8\pi}{3h^3}\int_0^{p_F} \frac{p^4c^2}{\sqrt{p^2c^2+m_e{}^2c^4}}\,dp$$

where p_F is the Fermi momentum, given by $p_F = \left(\frac{3h^3\rho}{8\pi\mu_e m_H}\right)^{\frac{1}{3}}$

Here we can have two extreme cases:

1. Non-relativistic case: We have,

$$\sqrt{p^2c^2+m_e{}^2c^4} \approx m_ec^2$$

   And after calculating the above integral we get the equation of state,

$$P = K_1\rho^{5/3}$$

[1]It contains the factor of mole fraction of hydrogen in the star, but that can be taken be approximated as a constant for a good deal of white dwarfs.

2. Fully-relativistic case: We have,

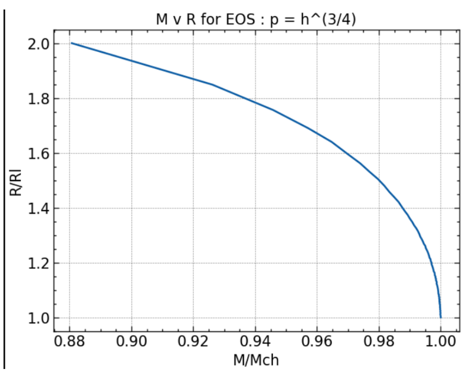
$$\sqrt{p^2c^2 + m_e{}^2c^4} \approx pc$$

And after calculating the integral we get the equation of state,

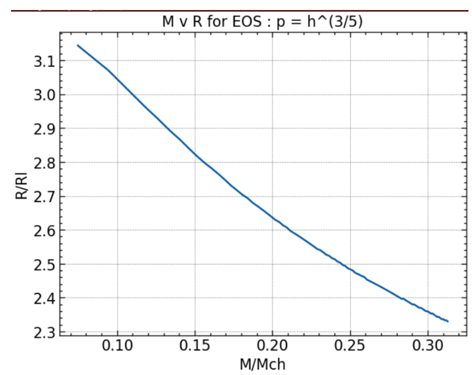
$$P = K_2\rho^{4/3}$$

Where K_1 and K_2 are some constants

Now, we can use these equations of state to solve the stellar structure equations and get a plot of Pressure v/s radial distance for these two extreme cases and to find Chandrasekhar mass limit this way we see the data from the plot of the fully-relativistic case. As shown in the graph below, we can see the mass getting constant and give us the value of Chandrasekhar mass limit.



(a) M v/s R curve for relativistic case, where *Rl* is the smallest radius obtained



(b) M v/s R curve for the non-relativistic case, where *Rl* is the smallest radius obtained

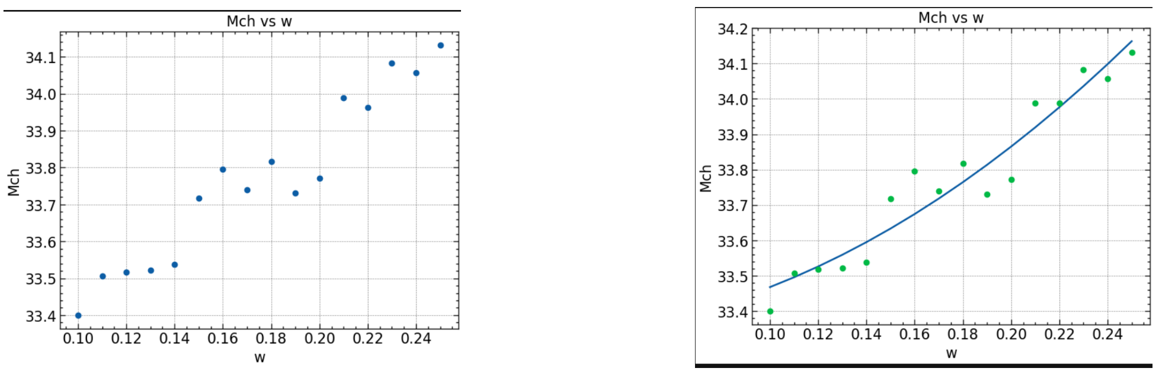
Figure 1: The M v/s R curves for the two extreme cases

Also, for non-extreme cases what we can do is fit a polytropic relation in a fixed number of intervals (say 100-200) and get multiple equations of states, which enables use to solve the structure equations for the general case.

3.2 The rotating case

So for the rotating case what we did was numerically solve the equation of structure for a rotating star with the equation of state of the fully relativistic case as that is where we get

the Chandrasekhar mass limit.
This process was done over a range of pressures to get multiple values of masses for a given value of rotation factor (w). Then taking the average of all these masses we obtained the value for the mass limit for this particular 'w'.
Then we repeated this for different values of 'w', the values of w were taken from 0.1 to 0.25 and we plotted the mass limit with respect to 'w'. The plot was then fitted with a quadratic.
The plot is given below,



(a) Unfitted (b) Fitted with quadratic

Figure 2: M_{ch} v/s w plot

Results

As we can see from the plot given above, the chandrasekhar mass limit increases as we increase the rate of rotation (w).
Also after fitting a quadratic to the data point we see that this is a decent fit, although we don't have a lot of data points and there are some errors because the solution of the stellar structure equation is not very accurate, but this fit seems to work well with these data points.
The data was fitted with $y = a + bx^2$ and we got the final fit as follows:

$$M_{Ch} = M_0 + \zeta w^2$$

$$M_0 = 33.335 \pm 0.001$$

$$\zeta = 13.428 \pm 0.076$$

4 Ring of Maximum Density

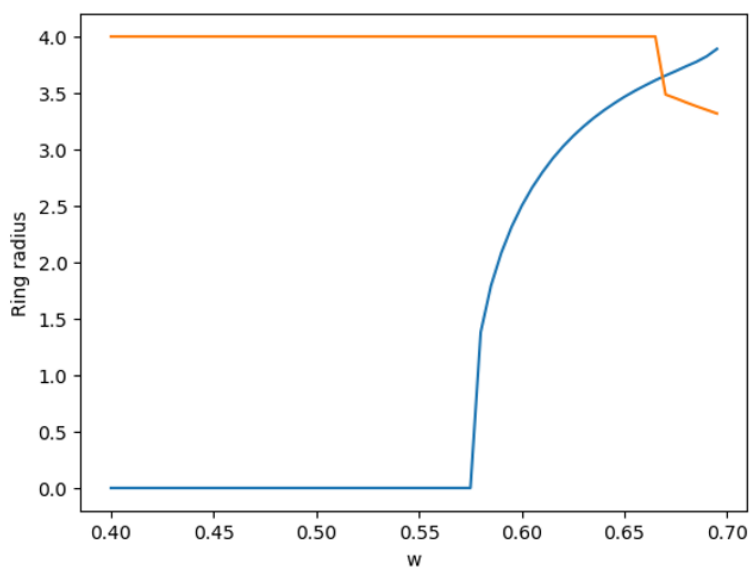
For non-rotating and slowly rotating stars, the maximum of pressure occurs at a single point, the centre. However, for rapidly rotating stars, the maximum of pressure occurs not at the centre, but rather in a ring around the centre, lying in the equatorial plane. The radius of the ring grows with increasing w. This change from a point into a ring is sharp and occurs after a critical value of w is reached. Mathematically, at the critical value, the central point turns from a maxima into a minima. So at centre, $\frac{\partial^2 p}{\partial \mu^2}$ becomes positive. Substituting the approximation for small ξ, we obtain the

$$w_c = \sqrt{(\frac{h(p_c)}{3})}$$

At $p_c = 1$, $h(p_c) = 1$. So

$$w_c = \sqrt{(\frac{1}{3})} \approx 0.578$$

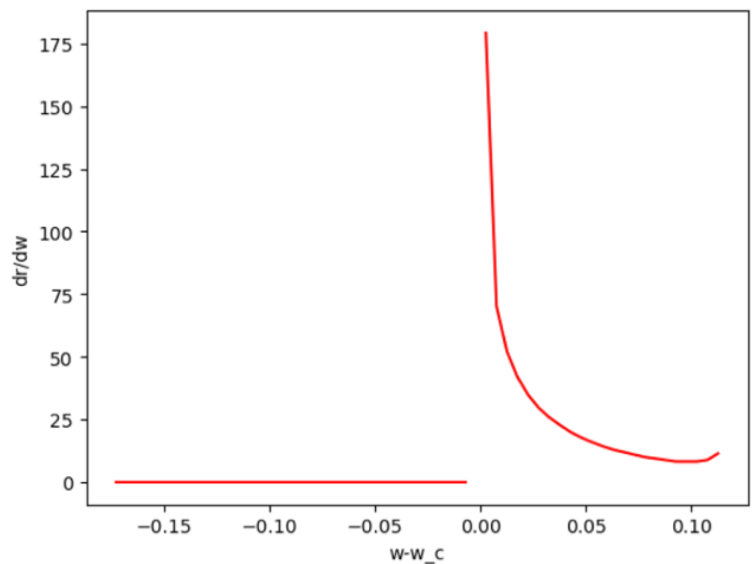
The star is solved for same value of p_c but different values of w and the ring radius-w curve was obtained as follows. To resolve the problems in computations mentioned before, a polynomial was fitted onto the values of $\frac{\partial p}{\partial \mu}$. It was found that for a good range of values of w, the error in the polynomial fitting was very small but increases with ξ. The Orange line indicates the value of ξ where the error in the approximation exceeds 0.01 (set to 4 if it is less than 4, to see the curve).



This formation of a ring of maximum density is essentially a type of mechanical phase transition. An important thing characterising the transition is the divergence of $\frac{dR}{dw}$, where R is the radius of the ring. In ordinary phase transitions, it is the latent heat that diverges. In our case it is $\frac{dR}{dw}$. In the right neighbourhood of w_c, the following approximation holds:

$$\frac{dR}{dw} = (w - w_c)^{-\beta}$$

where the exponent, beta giving the degree of divergence is an order parameter.



The value of beta was numerically obtained to be close to 0.5.