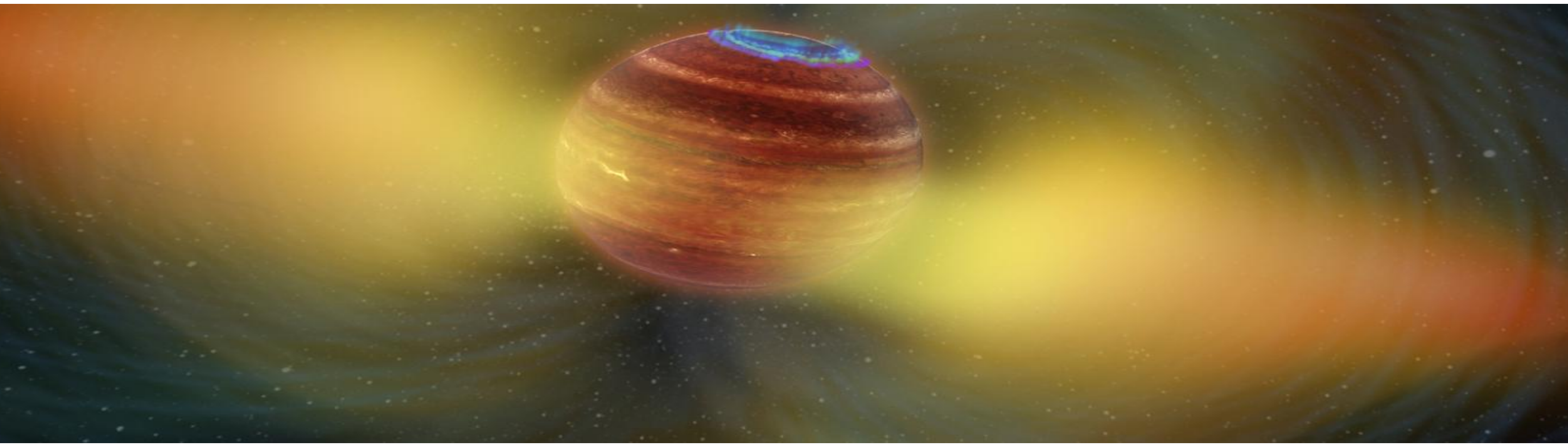


Krittika Summer Project

Solar System Dynamics Aditya Dungrani

August 5, 2023

Abstract

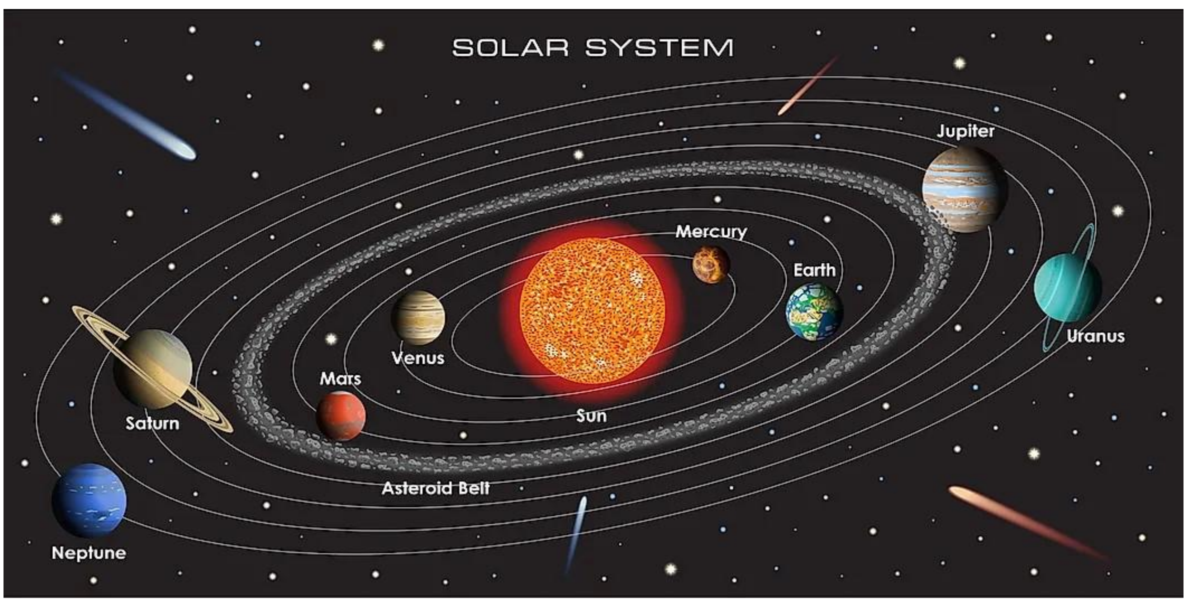
This report contains my thoughts, theories and simulations of the topics that I covered during the Krittika Summer Project. The subject is Solar System Dynamics where I covered a lot of interesting topics and phenomenons. Finally, I would like to thank my mentors **Adarsh Reddy** and **Dhananjay Raman** for their valuable guidance throughout the project.

Content

1. Celestial Mechanics

1.1 Introduction

In our Solar System, all the planets revolve around the sun in an elliptical manner. The sun, along with the entire solar system, orbits around the center of the Milky Way at an average speed of about 220 kilometers per second. The Milky Way galaxy itself is also moving in relation to other galaxies in the universe. The universe is constantly expanding, and galaxies are moving away from each other due to the overall expansion of space. The motion of the sun and our solar system is just a small part of the grand cosmic dance happening throughout the universe. Quite complicated isn't it? We will mostly focus on the elliptical motions of the planets around the Sun considering the milky way stationary. Let us focus on the governing forces and equations in the next section.



1.2 Equations of motion

We will be deriving the equations of motion based on the assumption that the planets are point masses, which is a reasonable assumption, given the large distances between the sun and the planets. Therefore using the *Newton's Law of Gravitational Forces* , we obtain,

$$F = G\frac{mm'}{r^2}(\hat{r})$$

Further breaking down the forces into different components,

$$F_x = F\frac{x-x'}{r}$$

$$F_y = F\frac{y-y'}{r}$$

$$F_z = F\frac{z-z'}{r}$$

Using *Newton's Second Law,*

$$\frac{d^2x}{dt^2} = Gm\frac{x-x'}{r^3}$$

$$\frac{d^2y}{dt^2} = Gm\frac{y-y'}{r^3}$$

$$\frac{d^2z}{dt^2} = Gm\frac{z-z'}{r^3}$$

$$r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

This system of equations looks quite complicated to solve, therefore, we will be using Numerical Methods like Runge Kutta, Euler etc.

1.3 Integration technique

For the interest of time and length of the report, we will be covering the more accurate and widely used 4^{th} order Runge Kutta. Neither will we be covering the proof of the method.

In the fourth-order Runge-Kutta algorithm, the derivative is computed at the beginning of the time interval, in two different ways at the middle of the interval, and again at the end of the interval. The two estimates of the derivative at the middle of the interval are given twice the weight of the other two estimates. The algorithm for the solution of (3.52) can be written in standard notation as

$$k_1 = \Delta t * f(t_n, x_n)$$

$$k_2 = \Delta t * f(t_n + \frac{\Delta t}{2}, x_n + \frac{k_1}{2}$$

$$k_3 = \Delta t * f(t_n + \frac{\Delta t}{2}, x_n + \frac{k_2}{2})$$

$$k_4 = \Delta t * f(t_n + \frac{\Delta t}{2}, x_n + k_3)$$

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Using the same in Newton's Equations of motion:

$$\begin{aligned}
k_{1v} &= a(x_n, v_n, t_n)\Delta t \\
k_{1x} &= v_n \Delta t \\
k_{2v} &= a(x_n + \frac{k_{1x}}{2}, v_n + \frac{k_{1v}}{2}, t_n + \frac{\Delta t}{2})\Delta t \\
k_{2x} &= (v_n + \frac{k_{1v}}{2})\Delta t \\
k_{3v} &= a(x_n + \frac{k_{2x}}{2}, v_n + \frac{k_{2v}}{2}, t_n + \frac{\Delta t}{2})\Delta t \\
k_{3x} &= (v_n + \frac{k_{2v}}{2})\Delta t \\
k_{4v} &= a(x_n + k_{3x}, v_n + k_{3v}, t + \Delta t) \\
k_{4x} &= (v_n + k_{3x})\Delta t, \\
v_{n+1} &= v_n + \frac{1}{6}(k_{1v} + 2k_{2v} + 2k_{3v} + k_{4v}) \\
x_{n+1} &= x_n + \frac{1}{6}(k_{1x} + 2k_{2x} + 2k_{3x} + k_{4x}).
\end{aligned}$$

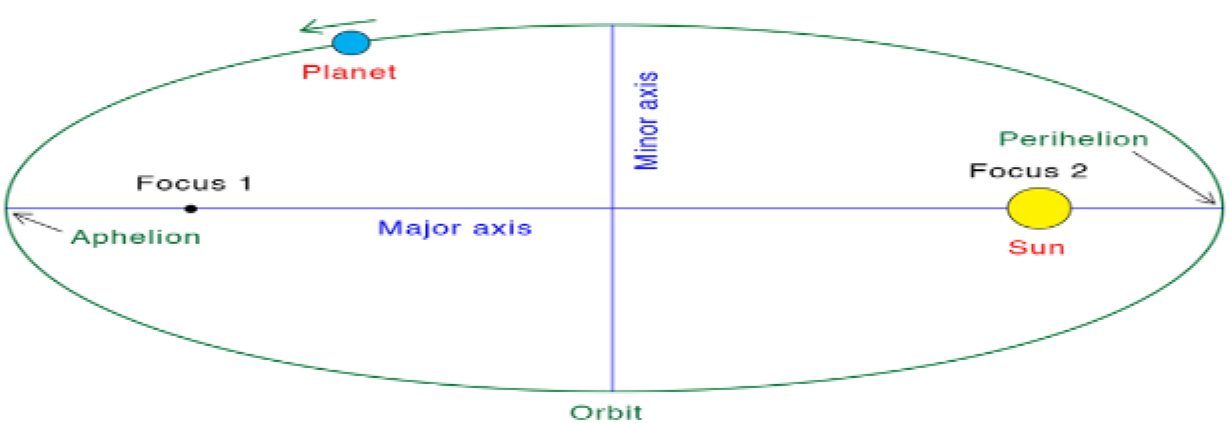
1.4 Kepler's Laws

Kepler's laws of planetary motion are a set of three fundamental principles formulated by the German astronomer Johannes Kepler in the early 17th century. These laws describe the motion of planets around the Sun and revolutionized our understanding of celestial mechanics. Let's have a look at all 3 Kepler's Laws one by one.

The man himself

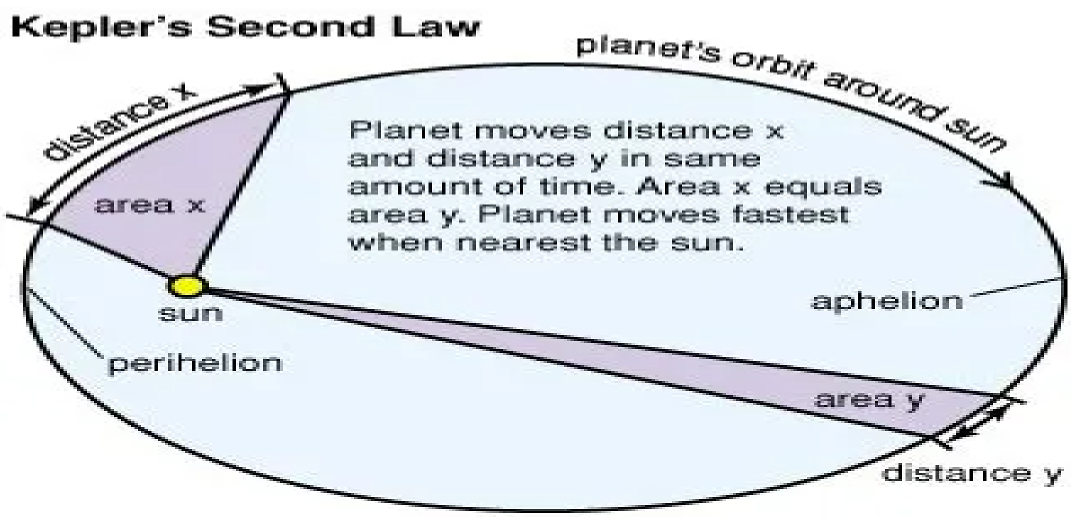
Kepler's First Law (Law of Ellipses)

Each planet orbits the Sun in an elliptical path, with the Sun at one of the two foci of the ellipse. In other words, the shape of a planet's orbit is not a perfect circle but an ellipse, where the Sun is located at one of the two fixed points known as foci. The distance between the foci determines the eccentricity of the orbit. If the eccentricity is close to zero, the ellipse becomes nearly circular.



Kepler's Second Law (Law of Equal Areas)

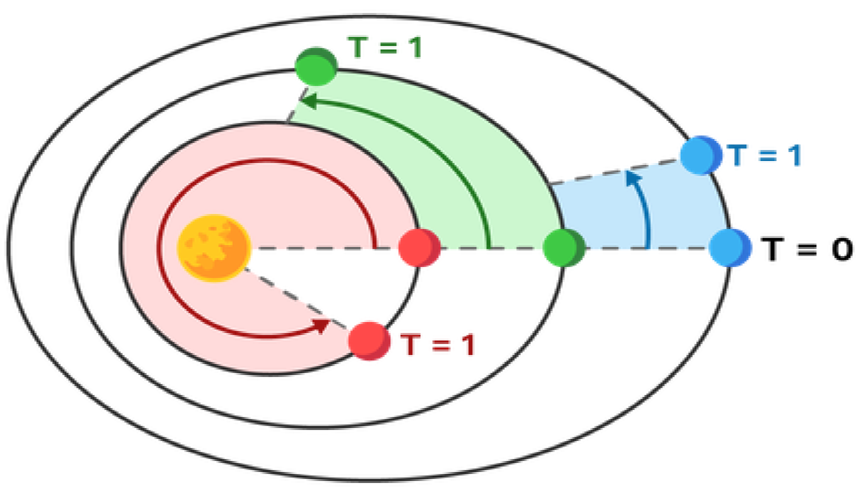
A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. This law means that a planet moves faster when it is closer to the Sun (perihelion) and slower when it is farther away (aphelion). As a result, a planet covers equal areas in its orbital path in equal periods of time.



Kepler's Third Law (Law of Harmonies)

The square of the orbital period (the time it takes for a planet to complete one orbit around the Sun) is directly proportional to the cube of the semi-major axis of its orbit (the average distance from the Sun to the planet).

Mathematically, this can be expressed as: $T^2 \propto a^3$, where T is the orbital period and a is the semi-major axis of the orbit. This law allows us to compare the orbital periods and distances of different planets. For example, a planet twice as far from the Sun as another planet will take approximately $2^{(2/3)} = 1.59$ times longer to complete one orbit.



1.5 Coordinate Systems

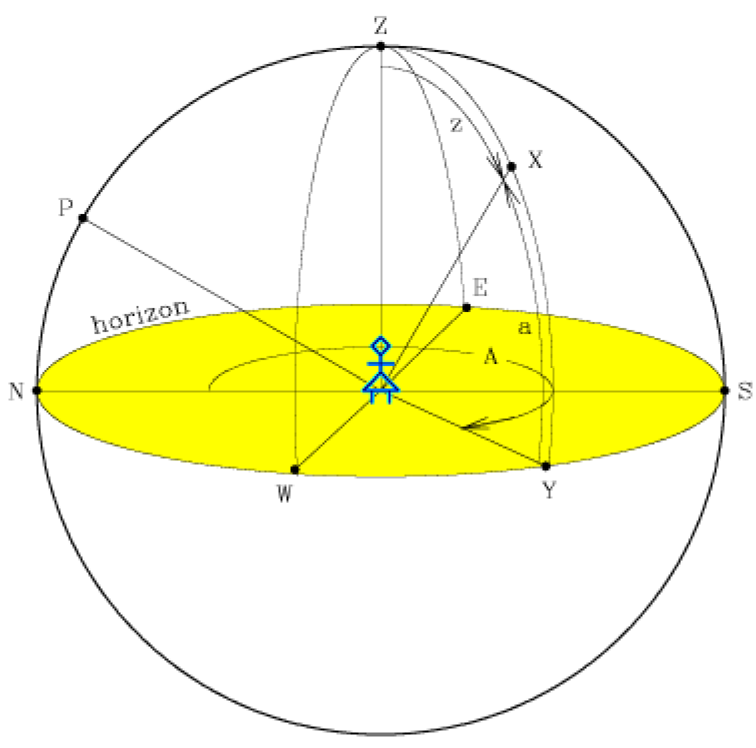
Now that we know a bit about different governing equations and laws about our Solar System, we must learn the different Coordinate Systems, each useful in different scenarios. We will be looking at relevant frames and motions in this section.

1.5.1 Horizontal Coordinate System

The Horizontal (alt-az) coordinate system is the simplest coordinate system, as it is based on the observer's horizon. The celestial hemisphere viewed by an observer on the Earth is shown above. The great circle through the zenith Z and the north celestial pole P cuts the horizon NESYW at the north point (N) and the south point (S). The great circle WZE at right angles to the great circle NPZS cuts the horizon at the west point (W) and the east point (E). The arcs ZN, ZW, ZY, etc, are known as verticals.

The two numbers which specify the position of a star, X, in this system are the azimuth, A, and the altitude, a. The altitude of X is the angle measured along the vertical circle through X from the horizon at Y to X. It is measured in degrees. (An often-used alternative to altitude is the zenith distance, z, of X, indicated by ZX. Clearly, z = 90 - a.) Azimuth is defined as the angle between the vertical through the north point and the vertical through the star at X, measured eastwards from the north point along the horizon from 0° to 360°. This definition applies to observers in both the northern and the southern hemispheres.

The main advantages of the alt-az system are that it indicates clearly how high a star is above the horizon and in what direction it can be found. The main disadvantage is that it is a local coordinate system, so two observers at different points on the Earth's surface will measure different altitudes and azimuths for the same star at the same time. In addition, a star's alt-az coordinates change with time as the celestial sphere appears to rotate. Despite these problems, most modern research telescopes use alt-az mounts, owing to their low cost and great stability.

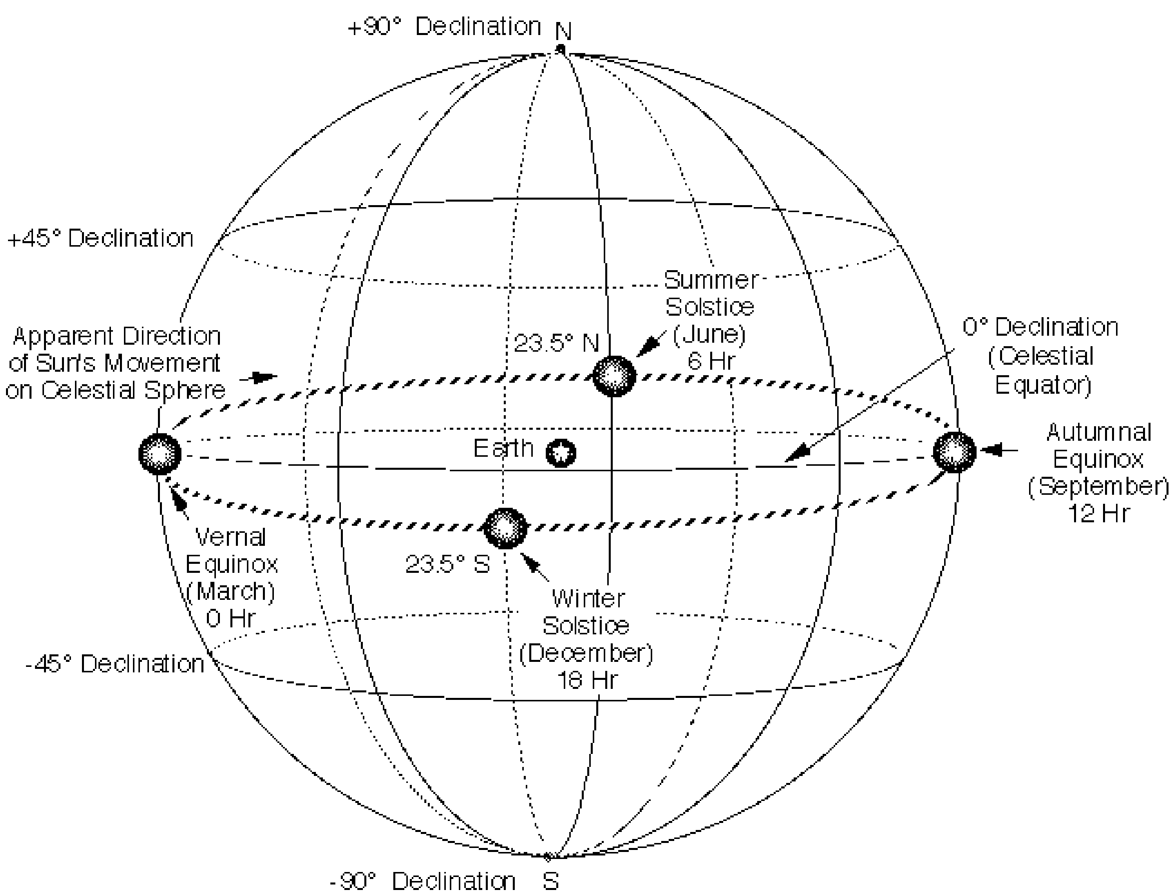


1.5.2 Equatorial Coordinate System

In the equatorial coordinate system, Earth's equator is the plane of reference. Earth's axis of rotation points to the north and south celestial poles. The celestial sphere is as large as the known universe, and Earth is at the center of this sphere. The celestial poles do not move as Earth rotates. For an observer standing at Earth's equator, the celestial poles are on opposite horizons at exactly the north and south points, and the celestial equator passes overhead going exactly from the east to west horizon.

To describe an object's location in the sky, we imagine a great circle through the celestial poles and the object and call this the object's hour circle. Where the sun in its path crosses the celestial equator each year around March 21 is called the vernal equinox, and it is this line that is the reference for the east-west coordinate of the object. Because Earth's axis is inclined 23.5° to the plane of its orbit around the sun, objects in the solar system (such as the planets and, from our perspective, the sun) move across the celestial sphere not along the equator, but rather in their own orbits, most of which are in nearly the same plane as Earth's orbit. This imaginary path of the sun's apparent motion, called the ecliptic, is a curving line that runs around the sphere ranging between 23.5° north and 23.5° south. The constellations of the zodiac all lie on the ecliptic. The object's elevation above the celestial equator is called declination

Equatorial Coordinate System



The coordinates of the object, then, are given by its declination and its right ascension or hour angle between the object's hour circle and the vernal equinox. Declination is expressed in degrees (0° to +90° if north of the equator, 0° to -90° if south of the equator). Right ascension is expressed either in degrees (0° to 360° measured eastward from the vernal equinox) or, more commonly, in hours, minutes, and seconds of time (0 to 24 hours). Declination and right ascension are absolute coordinates regardless of the observer's location on Earth or the time of observation. The only exception is due to the slight 26,000 year cyclic change in the equatorial coordinates because of the precession of Earth's axis. Precession causes the stars to appear to shift west to east at the rate of .01 degree (360 degrees/26,000 years) with respect to the vernal equinox each year. For example, in the time of ancient Rome, the vernal equinox was located in the constellation of Aries. It is now moving into the constellation of Aquarius.

There are more coordinate systems like Elliptical and Galactic, we shall look at them when we use them.

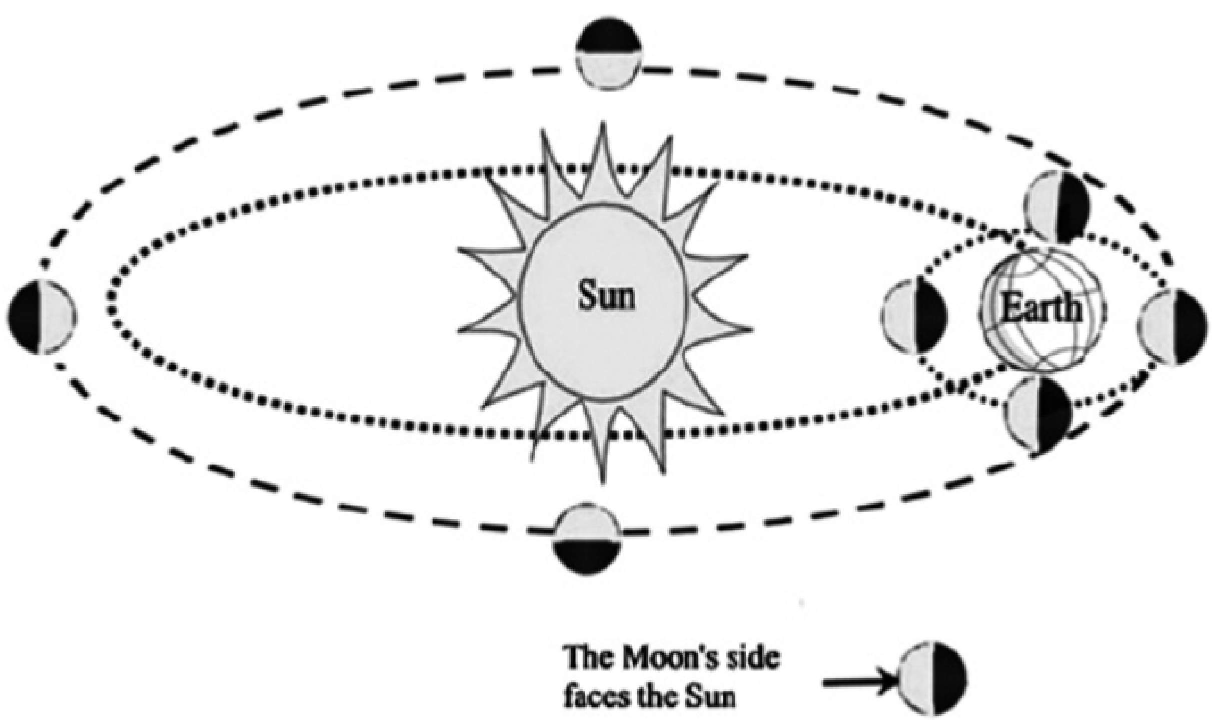
2. Orbits

Now that we have an idea about the motion, frames, coordinates and laws governing planetary motion, we will try to understand and simulate some interesting orbits and phenomena in our Solar System.

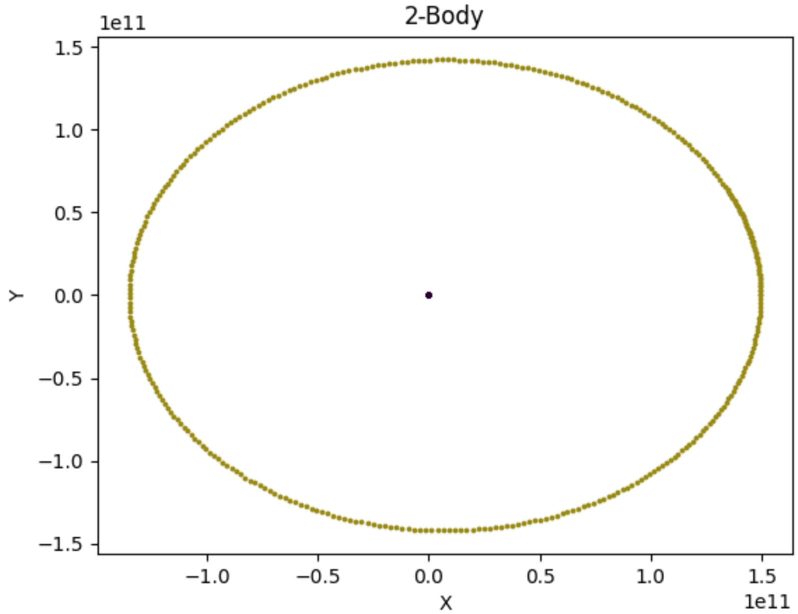
2.1 Sun-Earth System

The two-body problem is interesting in astronomy because pairs of astronomical objects are often moving rapidly in arbitrary directions (so their motions become interesting), widely separated from one another (so they will not collide) and even more widely separated from other objects (so outside influences will be small enough to be ignored safely).

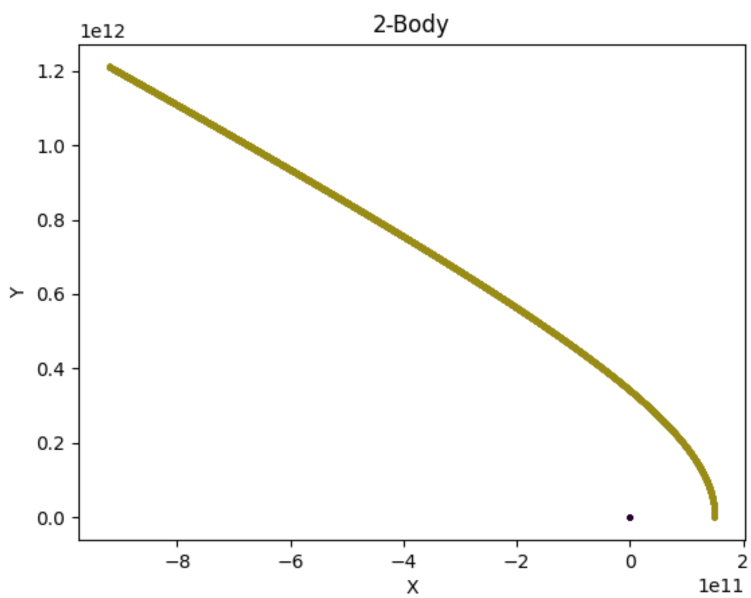
In the case of the Sun-Earth system, the Sun's much greater mass compared to Earth results in the Earth orbiting the Sun. Both the Sun and Earth experience a gravitational pull toward each other, causing the Sun to wobble slightly due to the gravitational influence of Earth, though this effect is negligible given the Sun's immense mass.



Considering the Sun and Earth as point masses, given the large distance between them, we numerically integrate Newton's Laws of Gravitation to simulate the Earth's motion around the sun.



True Sun - Earth system
T = 365 days



Sun - Earth system , if the earth had velocity > escape velocity
T = 365 days

We will discuss the code for this problem in the next section where we generalize it for n bodies.

2.2 N-body Simulation

In an N-body simulation, each celestial body (e.g., stars, planets, or galaxies) is treated as a point mass, and the simulation calculates the positions and velocities of these bodies over time as they gravitationally interact with one another. The code that I created for generalizing n body simulation :

```
def getAcc(pos ,mass, n): ## returns (n,3) acceleration

## getting x , y , z separately
x = pos[:, 0:1].reshape(n,1)
y = pos[:, 1:2].reshape(n,1)
z = pos[:, 2:3].reshape(n,1)

## relative coordinates Xij
x_rel = x.T - x
y_rel = y.T - y
z_rel = z.T - z

## r^3
r3 = x_rel**2 + y_rel**2 + z_rel**2
## r^(-1.5) keeping in mind rii is 0
r3[r3!=0] = r3[r3!=0]**(-1.5)

ax = G * (r3 * x_rel) @ mass
ay = G * (r3 * y_rel) @ mass
az = G * (r3 * z_rel) @ mass
return np.hstack((ax , ay , az))
```

The above function calculates and returns the acceleration of every planet using the position of the planets with respect to each other.

```
def solve(mass , pos , vel, n , t_final , dt):
# Parameters
t = 0
total_steps = np.int64(np.ceil(t_final/dt))

# Initial acceleration
acc = getAcc(pos , mass , n)

# Position matrix to save all positions (Number of bodies , 3 , total_time_frames)
pos_m = np.zeros((n,3,total_steps+1) , dtype=np.float64)
pos_m[:, :, 0] = pos

# Calculate positions
for i in range(total_steps):

        pos += vel*dt + acc*dt*dt/2
```

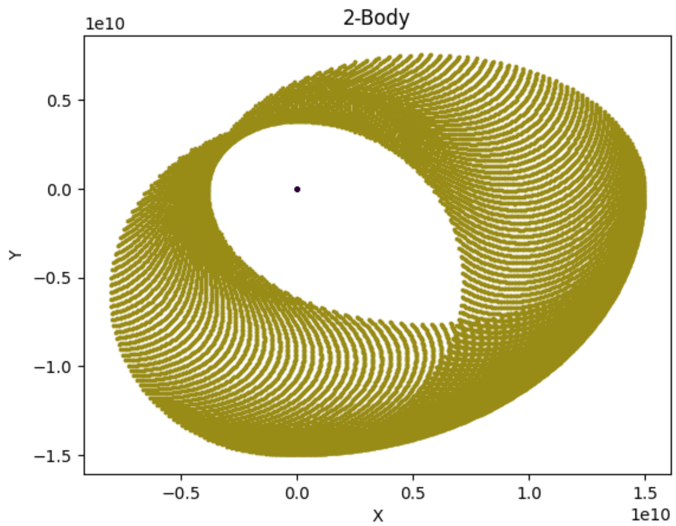
```
        vel += 0.5 * acc * dt
        acc = getAcc(pos , mass , n)
        vel += 0.5 * acc * dt
        t+=dt
        pos_m[: , : , i+1] = pos

return pos_m
```

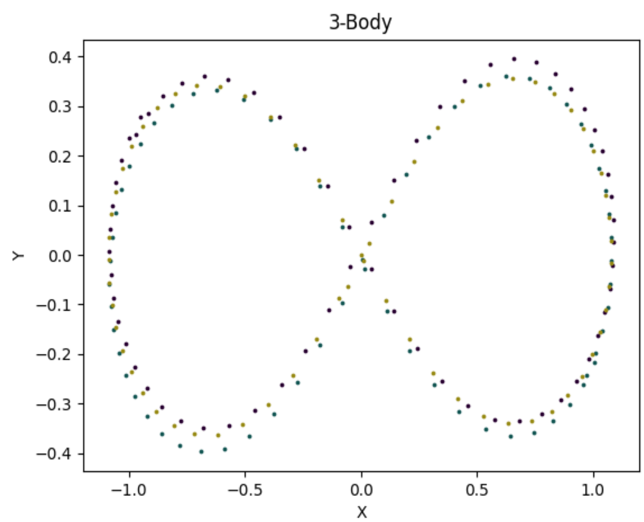
The above function uses euler method to solve for positions of each of the planets.

*Refer to my GitHub repository for the rest of the code

Here are some interesting stable orbits :



2-D donut



8-Orbit

2.3 Restricted 3-Body problem

Further let us dive into some special points like Lagrangian points, but before that let us try to understand the restricted 3-body problem, which will help us understand and simulate around the Lagrangian points.

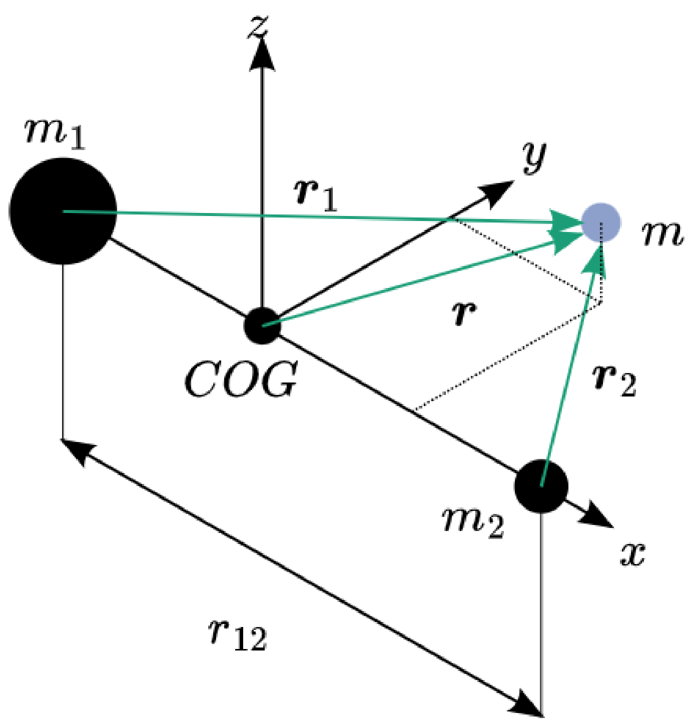
The restricted three-body problem is a simplified dynamical model used in celestial mechanics to study the motion of a small mass (e.g., a spacecraft or a small celestial body) in the gravitational field of two larger masses that orbit around their common center of mass. In this problem, the two larger masses are assumed to move in circular or elliptical orbits around their center of mass, while the small mass is considered to have a negligible influence on the motion of the two larger masses.

The three-body problem is inherently complex and does not have a general analytical solution, except for a few special cases. However, by introducing the "restricted" assumption that the small mass has a negligible effect on the motion of the larger masses, the problem becomes more tractable and can be approximated with greater ease. In the context of the restricted three-body problem, the motion of the small mass is governed by the gravitational forces exerted by the two larger masses, which are assumed to move in fixed orbits. The gravitational force from each of the two larger masses depends on the distance between the small mass and each of the larger masses. The equations of motion for the small mass can be derived using Newton's law of universal gravitation and the principles of classical mechanics.

Let's start the derivation, but first, some restrictions :

- There are two primary masses, and the mass of the tertiary object is extremely small in comparison to m1 and m2
- m1 > m2
- The two primary objects orbit in a circle around their center of mass

We first attach a non-inertial coordinate system to the barycenter of the system of and , such that the -axis of this coordinate system points towards . The distance from to is , which is also the radius of the circular orbit.



The inertial angular velocity of the reference frame is: (1)

$$\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{k}}$$

Where, (2)

$$\Omega = \frac{2\pi}{T}$$

T is the period of the orbit, and the orbital period for a circular orbit is: (3)

$$T = \frac{2\pi}{\sqrt{\mu}} r_{12}^{3/2}$$

where the gravitational parameter is given by: (4)

$$\mu = GM = G\left(m_1 + m_2\right)$$

we only need to find the x-coordinates, which we can do from the equation for the center of mass: (5)

$$m_1 x_1 + m_2 x_2 = 0$$

We need a second independent equation to solve for x_1 and x_2 . Fortunately, we know the distance between the masses is r_{12}. Solving for x_2 : (6)

$$x_2 = x_1 + r_{12}$$

To solve this set of equations, it’s convenient to define two dimensionless ratios: (7), (8)

$$\pi_1 = \frac{m_1}{m_1 + m_2} \qquad \pi_2 = \frac{m_2}{m_1 + m_2}$$

Also, (9), (10)

$$x_1 = -\pi_2 r_{12} \qquad x_2 = \pi_1 r_{12}$$

The position of the tertiary mass relative to the barycenter is: (11)

$$\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

The position of the tertiary mass relative to m_1 is: (12)

$$\boldsymbol{r}_1 = (x - x_1)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} = (x + \pi_2 r_{12})\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

and finally, the position of m relative to m_2 is: (13)

$$\boldsymbol{r}_2 = (x - \pi_1 r_{12})\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

Newton's second law requires the inertial acceleration. To find that, we first find the inertial velocity of m. We need to account for the rotating frame of reference. This means that the velocity and acceleration need to include the rotation of the coordinate system: (14)

$$\dot{\boldsymbol{r}} = \boldsymbol{v}_{\mathrm{COG}} + \boldsymbol{\Omega} \times \boldsymbol{r} + \boldsymbol{v}_{\mathrm{rel}}$$

where v_{cog} is the absolute velocity of the barycenter and v_{rel} is the velocity calculated in the moving coordinate system: (15)

$$\boldsymbol{v}_{\mathrm{rel}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}$$

Then we can find the absolute acceleration of m : (16)

$$\ddot{\boldsymbol{r}} = \boldsymbol{a}_{\mathrm{COG}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + 2\boldsymbol{\Omega} \times \boldsymbol{v}_{\mathrm{rel}} + \boldsymbol{a}_{\mathrm{rel}}$$

This equation can be simplified because we showed that the acceleration of the barycenter is zero for the two-body problem, a_{cog} = 0 . In addition, the angular velocity is constant since the orbit is circular, so $\Omega = 0$. Then, Eq. (56) can be simplified to:

Plugging everything in and simplifying: (17)

$$\ddot{\boldsymbol{r}} = \left(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x\right)\hat{\boldsymbol{\imath}} + \left(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y\right)\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}$$

For the tertiary body, the forces are due to both of the other masses: (18)

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}_1 + \boldsymbol{F}_2$$

The two forces are found by Newton's law of gravitation: (19) , (20)

$$\boldsymbol{F}_1 = -G\frac{m_1 m}{r_1^2}\hat{\boldsymbol{u}}_r\big)_1 = -\frac{\mu_1 m}{r_1^3}\boldsymbol{r}_1$$
$$\boldsymbol{F}_2 = -G\frac{m_2 m}{r_2^2}\hat{\boldsymbol{u}}_r\big)_2 = -\frac{\mu_2 m}{r_2^3}\boldsymbol{r}_2$$

Where,

$$\mu_1 = Gm_1 \qquad \mu_2 = Gm_2$$

And, (21) , (22)

$$\hat{\boldsymbol{u}}_r\big)_1 = \frac{\boldsymbol{r}_1}{r_1} \qquad \hat{\boldsymbol{u}}_r\big)_2 = \frac{\boldsymbol{r}_2}{r_2}$$

Combining equation 18, 19 and 20, and dividing by m, we get: (23)

$$\ddot{\boldsymbol{r}} = -\frac{\mu_1}{r_1^3}\boldsymbol{r}_1 - \frac{\mu_2}{r_2^3}\boldsymbol{r}_2$$

Finally, using (17) and (23),

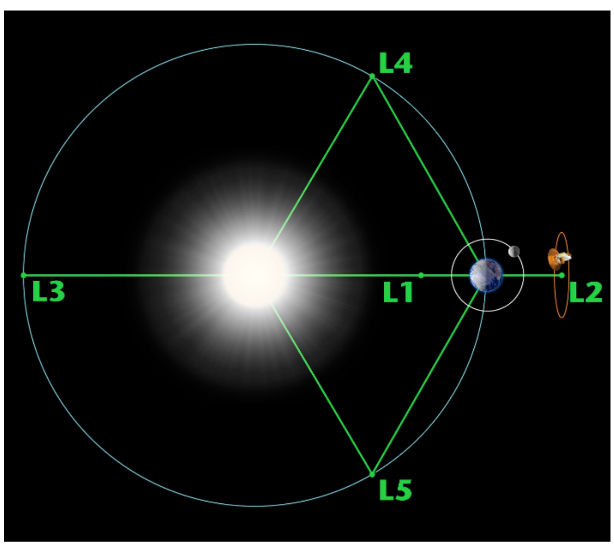
$$
\begin{aligned}
\ddot{x} - 2\Omega\dot{y} - \Omega^2 x &= -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \\
\ddot{y} + 2\Omega\dot{x} - \Omega^2 y &= -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \\
\ddot{z} &= -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z
\end{aligned}
$$

All the boring derivations are done and the stage is set for the next topic, lagrangian points.

2.4 Lagrangian points

Lagrangian points, also known as Lagrange points or libration points, are specific locations in space where the gravitational forces of two large celestial bodies, such as a planet and its moon or a planet and its star, create regions of gravitational equilibrium. In these regions, the gravitational pull from both bodies results in a stable or quasi-stable point where a smaller object, like a spacecraft or satellite, can remain relatively stationary with respect to the two larger bodies.

There are five Lagrangian points labeled L1 to L5, and they have specific geometric configurations in relation to the two larger masses.



Of the five Lagrange points, three are unstable and two are stable. The unstable Lagrange points – labeled L1, L2, and L3 – lie along the line connecting the two large masses. The stable Lagrange points – labeled L4 and L5 – form the apex of two equilateral triangles that have the large masses at their vertices. L4 leads the orbit of earth and L5 follows.

The L1 point lies on the line defined between the two large masses M1 and M2. It is the point where the gravitational attraction of M2 and that of M1 combine to produce an equilibrium. An object that orbits the Sun more closely than Earth would typically have a shorter orbital period than Earth, but that ignores the effect of Earth's gravitational pull. If the object is directly between Earth and the Sun, then Earth's gravity counteracts some of the Sun's pull on the object, increasing the object's orbital period. The closer to Earth the object is, the greater this effect is. At the L1 point, the object's orbital period becomes exactly equal to Earth's orbital period. L1 is about 1.5 million kilometers, or 0.01 au, from Earth in the direction of the Sun.

The L2 point lies on the line through the two large masses beyond the smaller of the two. Here, the combined gravitational forces of the two large masses balance the centrifugal effect on a body at L2. On the opposite side of Earth from the Sun, the orbital period of an object would normally be greater than Earth's. The extra pull of Earth's gravity decreases the object's orbital period, and at the L2 point, that orbital period becomes equal to Earth's. Like L1, L2 is about 1.5 million kilometers or 0.01 au from Earth (away from the sun). An example of a spacecraft at L2 is the James Webb Space Telescope, designed to operate near the Earth–Sun L2.

The L3 point lies on the line defined by the two large masses, beyond the larger of the two. Within the Sun–Earth system, the L3 point exists on the opposite side of the Sun, a little outside Earth's orbit and slightly farther from the center of the Sun than Earth is. This placement occurs because the Sun is also affected by Earth's gravity and so orbits around the two bodies' barycenter, which is well inside the body of the Sun. An object at Earth's distance from the Sun would have an orbital period of one year if only the Sun's gravity is considered. But an object on the opposite side of the Sun from Earth and directly in line with both "feels" Earth's gravity adding slightly to the Sun's and therefore must orbit a little farther from the barycenter of Earth and Sun in order to have the same 1-year period. It is at the L3 point that the combined pull of Earth and Sun causes the object to orbit with the same period as Earth, in effect orbiting an Earth+Sun mass with the Earth-Sun barycenter at one focus of its orbit.

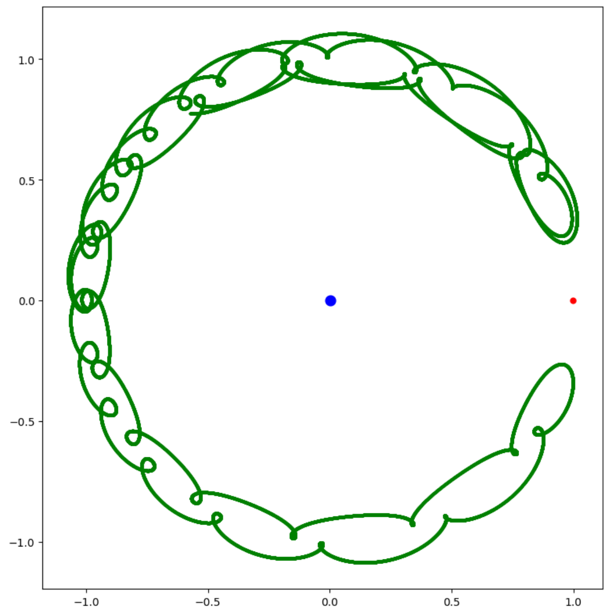
The L4 and L5 points lie at the third vertices of the two equilateral triangles in the plane of orbit whose common base is the line between the centers of the two masses, such that the point lies 60° ahead of (L4) or behind (L5) the smaller mass with regard to its orbit around the larger mass.

Now that we know more about the Lagrange points and their location, let’s try to simulate and see what orbit a satellite in different lagrangian points follow.

2.5 Horseshoe and Tadpole orbit

Horseshoe Orbit:

When the third object is extremely close to the L4 and L5 points, it demonstrates a quasi-periodic orbit around that point.One of the examples is horseshoe orbit. In a horseshoe orbit, the smaller object orbits around one of the larger masses but experiences a gravitational interaction that causes it to oscillate in a shape resembling a horseshoe. It appears as if the smaller object is approaching the larger mass, then changing direction, and moving away, only to repeat the process again. The horseshoe orbit occurs around the L3 Lagrangian point in the Sun-Earth system. Here, the smaller object orbits the Sun slightly faster or slower than the Earth, and the gravitational interaction between the two causes the peculiar horseshoe-like motion. It's as if the smaller object is "chasing" the Earth around the Sun. Here is an example I simulated :

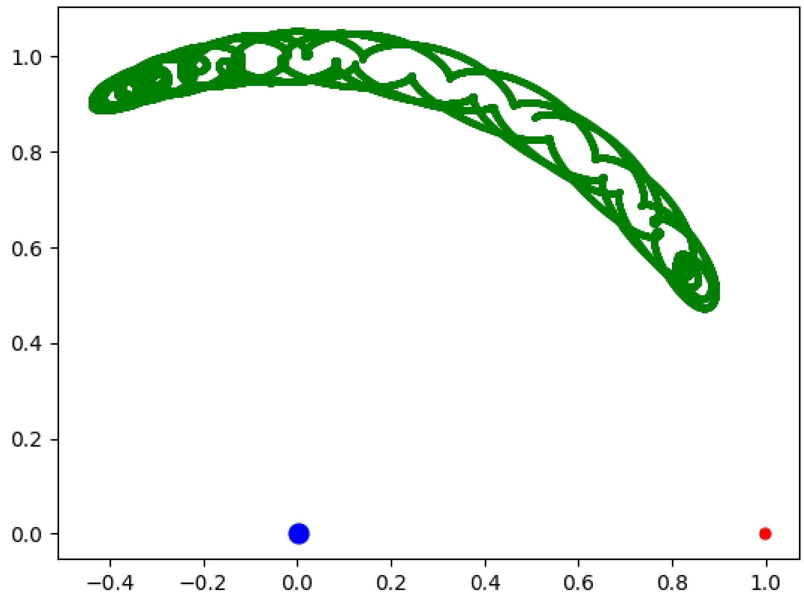



```
u2 = 0.001
x = 0.5 - u2 + 0.0065
y = sqrt(3) / 2 + 0.0065
xd = 0
yd = 0
```

Horseshoe Orbit

Tadpole Orbit:

In a tadpole orbit, the smaller object also orbits one of the larger masses but does so in a long, elongated shape resembling a tadpole. It appears as if the smaller object is moving ahead of the larger mass, then looping back, and moving ahead again in a repeating pattern. The tadpole orbit occurs around the L4 and L5 Lagrangian points in the Sun-Jupiter system (or other planet-moon systems). These points are associated with the formation of Trojan asteroids, which gather around these Lagrangian points due to their gravitational interactions with the larger bodies. The tadpole-shaped orbit of these asteroids is a result of the balance between the gravitational forces of the two larger masses.



Tadpole Orbit

μ2 = 0.000953875
x = -0.97668
y = 0
xd = 0
yd = -0.06118

3. Celestial Phenomena

3.1 Analemma

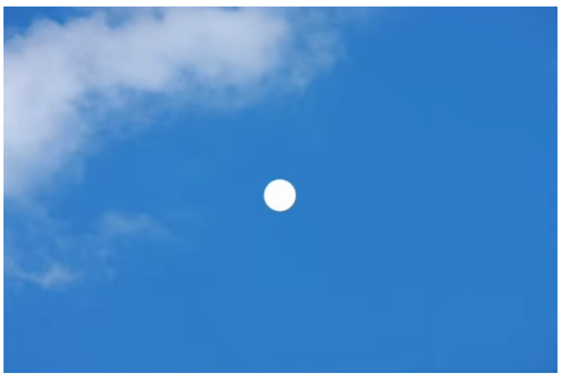
An Analemma is the diagram showing the position of the Sun in the sky, as seen from a fixed location at the same time, throughout the year. Its structure looks similar to that of the figure: "8". But unlike the figure of 8, the two loops of the Analemma are unequal in size with one of them larger than the other. In astronomy, the Analemma is considered one of the most difficult and demanding phenomena to imagine because it is never present all at once. It requires a virtual image made at the same time of day on 30 to 50 days throughout the year. The below photo was taken in 1998–99 of Analemma from the office window of Bell Labs, Murray Hill, NJ.

There are two factors that contribute to the formation of the Analemma pattern and they are completely independent of each other.

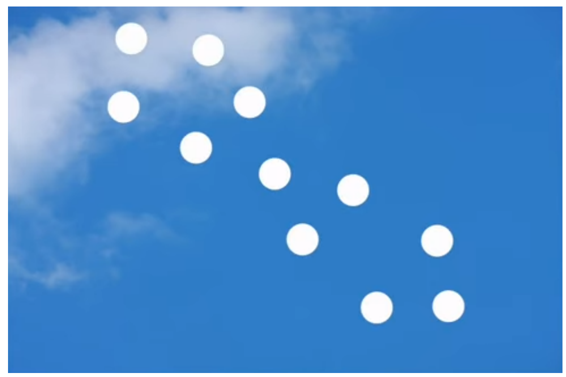
- Earth revolves around the Sun in an elliptical orbit.
- The Earth is tilted on its axis 23.5° in relation to the plane of its orbit around the Sun.

Let's take four cases to understand the factors better.

No tilt
Circular Orbit

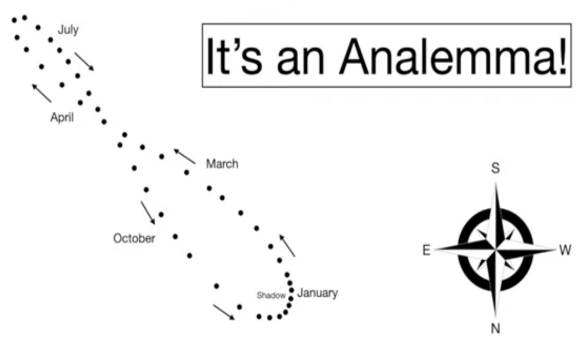


Big tilt
Elliptical Orbit



No tilt
Elliptical Orbit

Big tilt
Circular Orbit



I tried simulating the analemma on earth, but there is some mathematical error, which is giving me a symmetrical Analemma which should not be the case considering the tilt and orbit of earth. Code can be found [here]

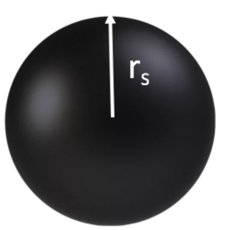
3.2 Black hole

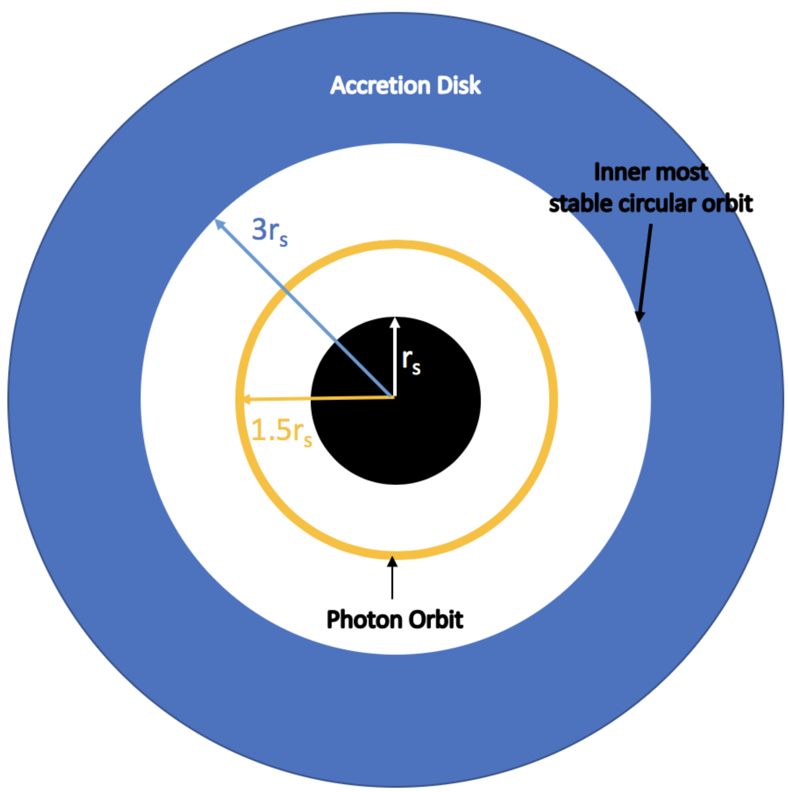
Let us now try to understand some interesting phenomenons related to black holes. But first let's try to understand what a black hole is. Well, a black hole is simply an astronomical object that is formed when a massive star collapses under its own gravitational force. When a star exhausts its nuclear fuel, it can no longer support its own weight against gravity, leading to a catastrophic implosion. The core of the star collapses to an incredibly dense and compact state, creating a region in space where gravity is so strong that nothing, not even light, can escape from it. This point of infinite density and zero volume is called a singularity.

The boundary around the singularity is known as the event horizon. Once an object crosses the event horizon of a black hole, it is trapped, and there is no known way for it to escape from the black hole's gravitational pull.

A photon orbit, also known as a photon sphere, is a region around a black hole where photons (particles of light) can travel in closed, stable orbits. In this region, the gravitational pull of the black hole is so strong that photons can be trapped and forced to travel in a circular path around the black hole, without falling into it or escaping from it. The photon sphere is located at a distance from the black hole that is approximately 1.5 times the Schwarzschild radius.

A black hole is also appeared to have a thin band around it made of all the stellar debris, dust and matter that was passing through the event horizon and this band of matter which is at the edge of the horizon and has not fallen into the black hole is known as an accretion disk. It is located at a distance 3 times the Schwarzchild radius.





The Schwarzchild radius is given by :

$$r_{\mathrm{BH}} = \frac{2GM_{\mathrm{BH}}}{c^2}$$

Let's move on the simulate the path of photon around a black hole, we will use the following equation, where E is the energy of the system,

$$\frac{dr}{d\theta} = \sqrt{\frac{2E}{l^2}d^4 - d^2\left(1 - \frac{r_{\mathrm{BH}}}{d}\right)}$$

We also have, by conservation of momentum,

$$r^2\dot{\theta} = l$$

Using these equations, we can numerically solve the path of photon. Let's look at the code:

```
def get_L(R, thetad):
      return (R ** 2) * thetad

def get_E(R, Rd, l, rbh, thetad ):
      l = get_L(R, thetad)
      return 0.5 * (Rd ** 2) + (l ** 2) / (2 * (R ** 2)) - (l ** 2) * rbh
/ (2 * (R ** 3))
```

The above function helps me to calculate the energy and momentum at any time,

```
def thetadot(l,r):
      return l/(r**2)
def drdtheta(E,l,r,rbh):
      t = 2 * E - (l**2)/(r**2) * (1- rbh /r)
      if(t > 0):
            return - np.sqrt(t)
      else:
            return np.sqrt(-t)
```

These calculate the thetadot and dr/dt at every position.

```
def rk4(E, l, r, rd,rbh, theta, thetad, dt):

    k1 = dt * thetadot(l,r)
    l1 = dt * drdtheta(E,l,r,rbh)

    k2 = dt * thetadot(l, r+l1/2)
    l2 = dt * drdtheta(E,l,r+l1/2,rbh)

    k3 = dt * thetadot(l, r+l2/2)
    l3 = dt * drdtheta(E,l,r+l2/2,rbh)

    k4 = dt * thetadot(l, r+l3)
    l4 = dt * drdtheta(E,l,r+l3,rbh)

    return theta + ( k1 + 2*k2 + 2*k3 + k4)/6, r + ( l1 + 2*l2 + 2*l3 +
l4)/6
```

Results :