Solar System Dynamics

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Techniques Used

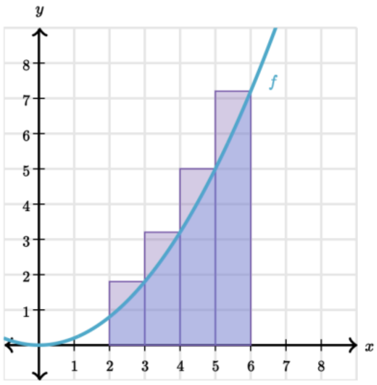
Euler Method of Numerical Integration

The Euler method is a simple numerical integration technique that approximates the solution of ordinary differential equations. It is a first-order method and involves using a forward difference formula to estimate the next state of a system. For a given initial position r_0 and velocity v_0, the Euler method iteratively calculates the next position r_n and velocity v_n using the following equations:

$$v_{n+1} = v_n + \frac{F(r_n, v_n)}{m}\Delta t$$

$$r_{n+1} = r_n + v_n \Delta t$$

where $F(r, v)$ is the force on the object as a function of object's position and velocity.



Euler Method is based on approximating an integral as a sum of rectangular areas. As we can see, it's not very precise, but gets more and more precise as we decrease the size of the time step.

Euler Richardson Method of Numerical Integration

The Euler-Richardson method, also known as the semi-implicit Euler method, is an improved version of the Euler method. It is a second-order method that provides better accuracy and energy preservation. The equations for the Euler-Richardson method are as follows:

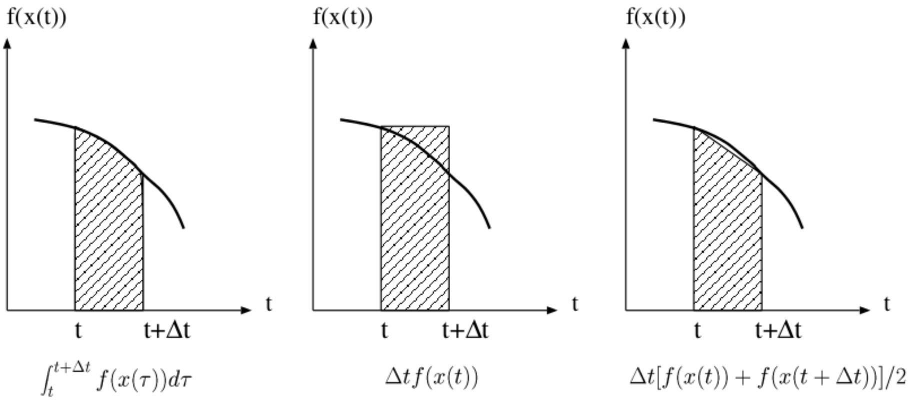
$$v_{mid} = v_n + \frac{1}{2} \cdot \frac{F(r_n, v_n)}{m} \Delta t$$

$$r_{mid} = r_n + \frac{1}{2} \cdot v_n \Delta t$$

$$v_{n+1} = v_n + \frac{F(r_{mid}, v_{mid})}{m} \Delta t$$

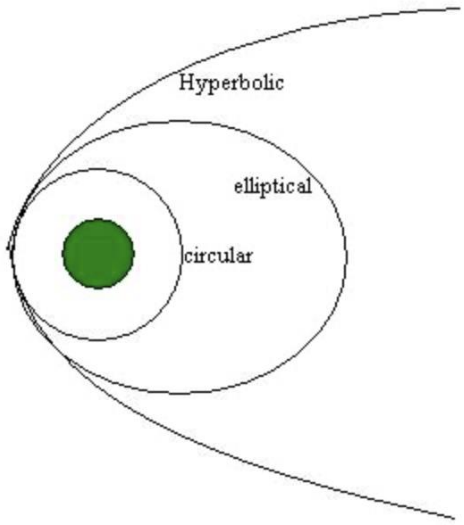
$$r_{n+1} = r_n + v_{mid} \Delta t$$

Euler Richardson Method is based on approximating the integral as a sum over trapeziods rather than rectangles. This gives us much better accuracy for the same time step, and hence is much more effective.



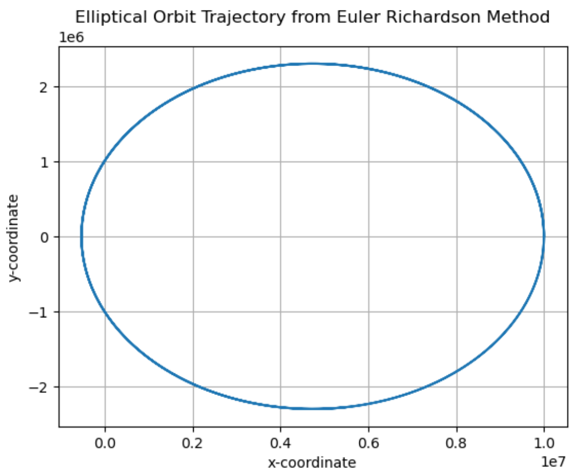
Celestial Orbits

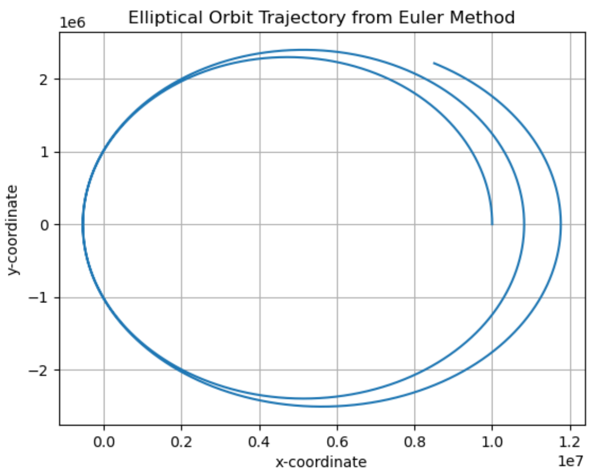
Celestial orbits are the graceful paths that celestial bodies, such as planets and comets, follow in space due to the gravitational forces between them. Two primary types of orbits are elliptical and hyperbolic. Elliptical orbits, characterized by an eccentricity between 0 and 1, exhibit periodicity around a central body. The total energy of an elliptical orbit is negative, with the kinetic energy changing as the distance varies. Hyperbolic orbits, on the other hand, have eccentricity greater than 1 and indicate escape trajectories from the central body. Their total energy is positive, with the kinetic energy increasing as distance increases. These orbits are governed by the polar conic equation, $r = \frac{p}{1+e \cdot \cos\theta}$, where r is the distance from the centre, p is the semi-latus rectum, e is the eccentricity, and θ is the polar angle. Understanding the nuances of these orbits and their associated energies is essential for comprehending the dynamics of celestial objects within our solar system and beyond.



Elliptical Orbit Simulations

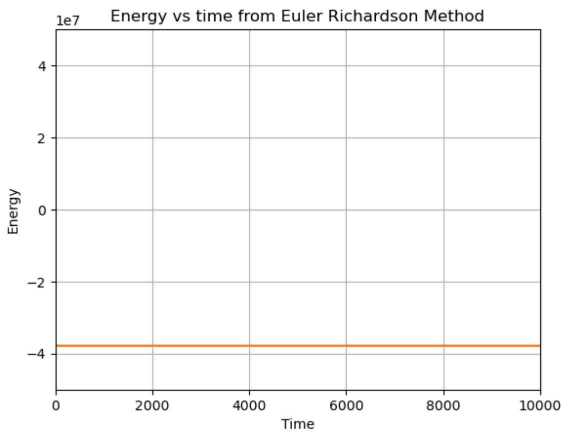
Using suitable initial conditions, we simulated elliptical orbit using both the Techniques mentioned above

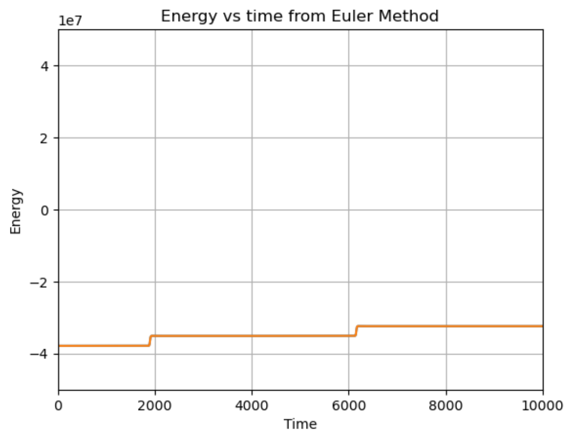




As we can see, Euler Richardson method gives much better accuracy in comparison to the Euler Method Plot

We can also verify this by looking at the Energy vs Time plots of both of both the simulations

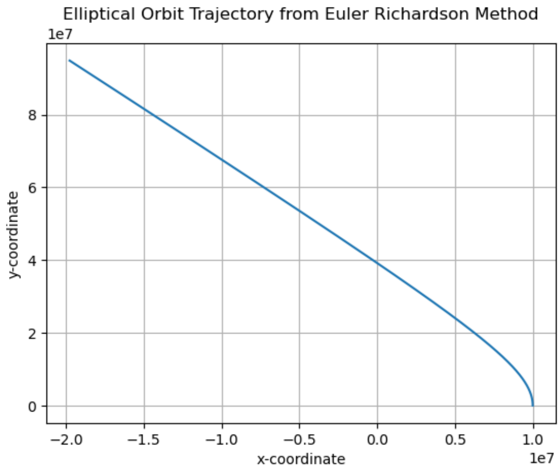


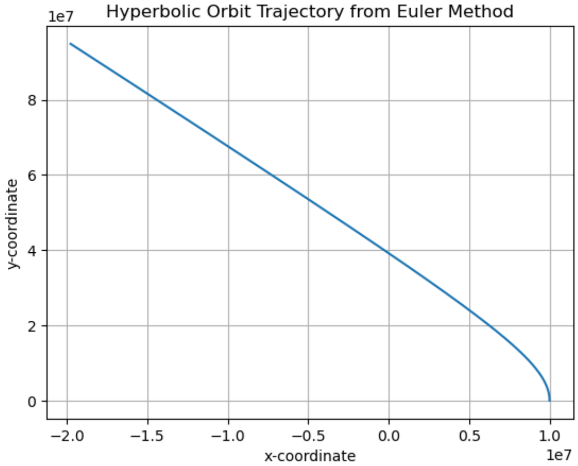


The energy is nearly constant in Euler Method, which is what we expect from theoretical calculations as well.

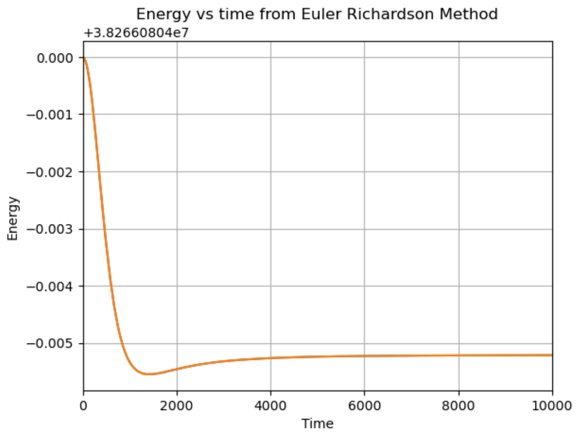
Hyperbolic Orbit Simulations

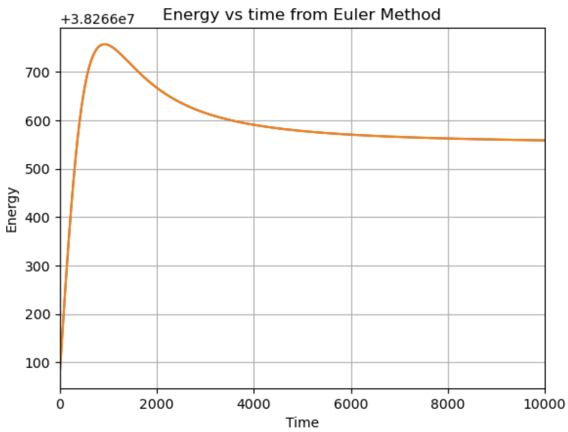
Similar to Elliptical Orbit Simulations, we use initial conditions and both the techniques to arrive at the following simulations





Unlike Elliptical Orbit simulations, we can't tell apart the 2 just by the plots, so we resort to Energy vs Time plots





As we can see, Euler method energy has variations on the order of 100, whereas Euler Richardson method energy has fluctuations on the order of 0.001, thus again making Euler Richardson much more accurate.

Reduced 3 Body System

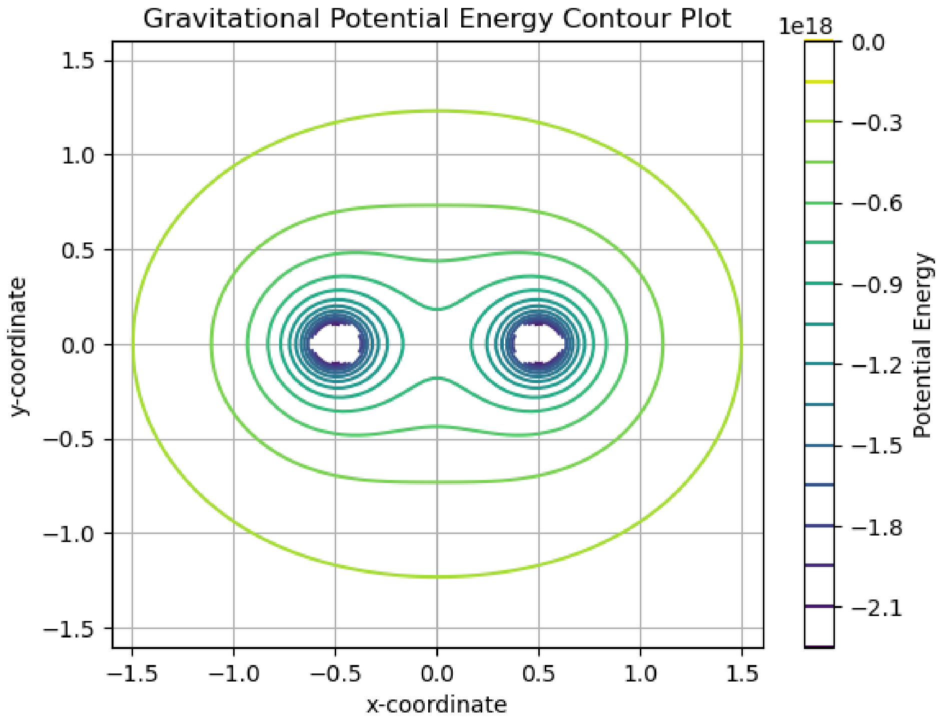
Three-body systems in celestial mechanics involve the complex interactions of three celestial bodies influenced by gravitational forces. These systems can encompass scenarios such as a planet, moon, and star or multiple stars orbiting each other. Analysing the motion and stability of such systems is challenging due to the intricate gravitational interactions and the absence of exact solutions in most cases. The study of three-body systems provides insights into celestial mechanics, planetary dynamics, and our understanding of the universe's intricate dance of gravitational forces.

A Reduced 3 Body System occurs when one of the 3 bodies is negligible in mass as compared to the other 2, and hence has little effect on the motion of the massive bodies.

We will attempt to plot the potential energy field of such a system as experienced by the small mass. The Equation of potential energy is:

$$V(r) = -\frac{G \cdot m \cdot M_1}{|r - r_1|} - \frac{G \cdot m \cdot M_2}{|r - r_2|}$$

where G represents the gravitional consta nt and represent the position vectors of the 3 masses involved

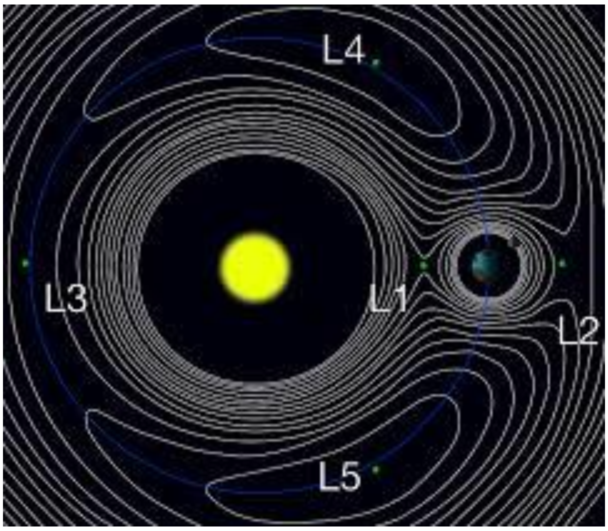


Lagrange Points in Solar System

Lagrange points are unique positions within a two-body or multi-body system where the gravitational forces of the participating bodies create points of equilibrium. In the context of the Sun-Earth system, five Lagrange points, labeled L_1 to L_5, exist.

L_1, L_2 and L_3 are situated along the line connecting the Sun and Earth. L_1 and L_2 are very close to Earth, whereas L_3 is diametrically opposite to earth and unstable in nature

L_4 and L_5 are located at equilateral triangles formed by the Earth and Sun, creating stable points that correspond to the apexes of equilateral triangles. These points have been associated with the Trojan asteroids, co-orbiting with a larger celestial body. These are also responsible for many interesting orbits like the Tadpole and Horseshoe orbits



Tadpole & Horseshoe Orbits

Tadpole orbits involve a small body orbiting one of the Lagrange points in a tadpole-like motion. As the larger body moves along its orbit, the smaller body appears to oscillate around the Lagrange point. This peculiar dance results from the interaction of gravitational forces and is a manifestation of the delicate balance between attractive and repulsive forces.

The equations representing the forces in such lagrange fields are:

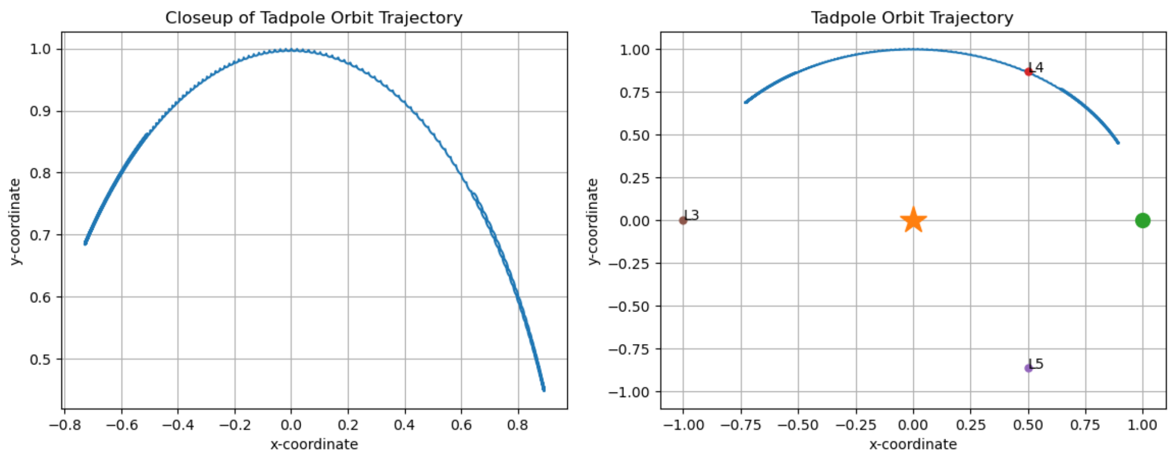
$$\frac{d^2x}{dt^2} - \frac{2dy}{dt} = \frac{\partial U}{\partial x}$$

$$\frac{d^2y}{dt^2} + \frac{2dx}{dt} = \frac{\partial U}{\partial y}$$

$$U = \frac{1}{2} \cdot (x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

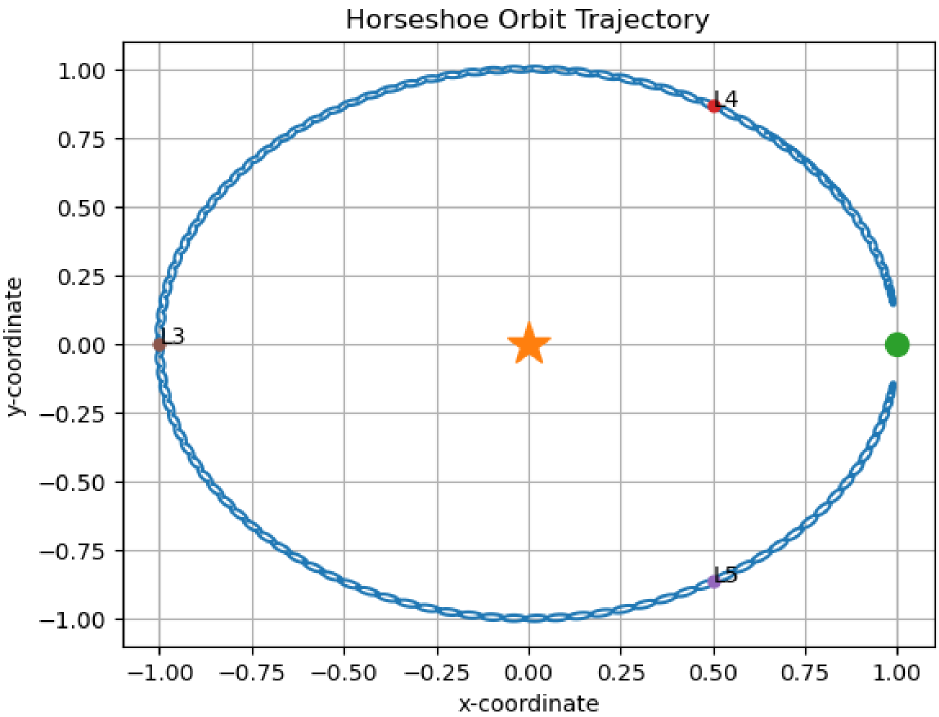
where U represents pseudo potential of the system and μ_1, μ_2 represents $\frac{m_1}{m_1+m_2}, \frac{m_2}{m_1+m_2}$

Using these Equations along with correct initial conditions (slight pertubation from L_4) and Euler Richardson Method, we get



Horseshoe orbits, on the other hand, occur when a small body orbits around a Lagrange point in a horseshoe-shaped path. This phenomenon arises due to the interplay of gravitational forces from both the larger and smaller bodies, causing the small body to move closer and farther from the Lagrange point in a cyclic manner.

Using the same equations and slightly different initial conditions, we get


Horseshoe Orbit Trajectory
L4
L3
L5
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x-coordinate
y-coordinate

Photons near a Black hole

A black hole is a remarkable cosmic entity born from the gravitational collapse of a massive star. It possesses an incredibly dense core, called the singularity, surrounded by an event horizon beyond which nothing, not even light, can escape due to the intense gravitational pull. This phenomenon arises from the extreme curvature of spacetime caused by the immense mass concentration, creating a gravitational well so profound that it distorts the very fabric of spacetime itself.

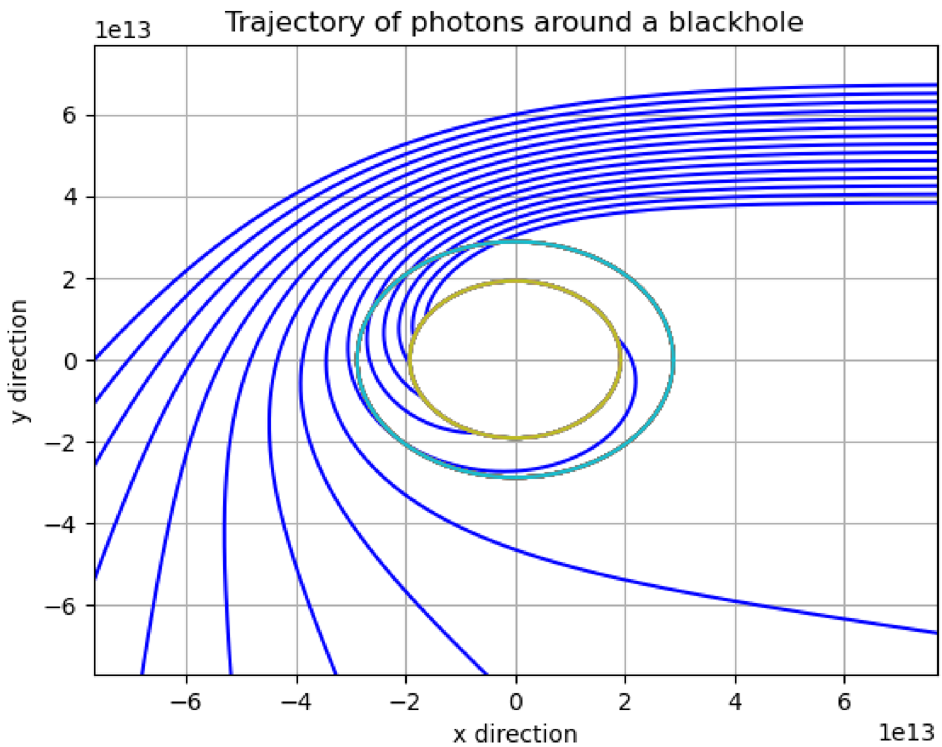
The colossal gravitational force exerted by a black hole profoundly affects the trajectory of photons, the particles of light. As photons pass close to a black hole, the curvature of spacetime around it causes their paths to warp, bending them from their originally straight trajectory. This phenomenon, known as gravitational lensing, is a consequence of Einstein's General Theory of Relativity. The extent of bending depends on the black hole's mass and proximity, resulting in the mesmerizing visual effect of light curving around the invisible gravitational behemoth.

We can model the trajectory of these photons using the energy equation:

$$E = \left(\frac{dr}{dt}\right)^2 + \frac{l^2}{2r^2} - \frac{l^2 r_{BH}}{2r^3}$$

where $r_{BH} = \frac{2GM_{BH}}{c^2}$ is the event horizion radius of the black hole and l is the angular momentum per unit mass.

Using this equation, we get the following plot



Analemma

An analemma is a captivating celestial pattern that results from the combined effects of Earth's axial tilt and its elliptical orbit around the Sun. This figure-eight-shaped curve traces the Sun's apparent position in the sky at the same local solar time throughout the year. The analemma captures the changing declination and right ascension of the Sun, creating a visual representation of how the Sun's position varies over different days. This celestial phenomenon offers a vivid illustration of the intricate interplay between Earth's axial tilt and orbital eccentricity, providing insights into the complex dynamics that shape our planet's relationship with the Sun across the changing seasons.

We can model the position of sun as seen from a planet's sky using spherical trigonometry

$$\delta = \arcsin(sin\varepsilon \cdot sin\lambda)$$

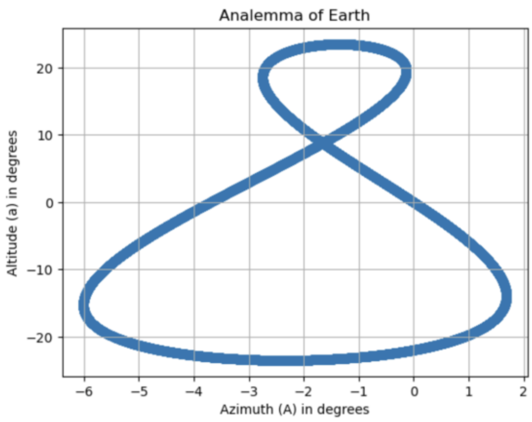
$$\alpha = \arccos\left(\frac{cos\lambda}{cos\delta}\right)$$

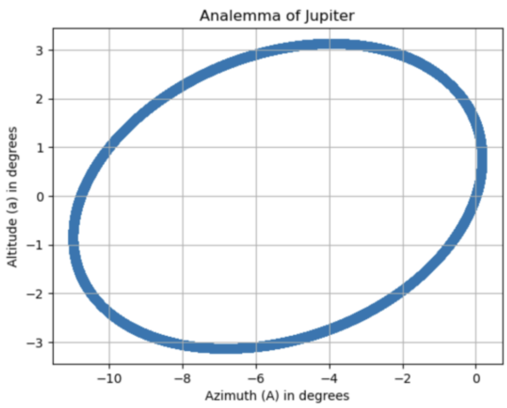
$$\int_{\theta_0}^{\theta} \frac{d\theta}{(1 + ecos\theta)^2} = \frac{2\pi t}{T(1 - e^2)^{\frac{3}{2}}}$$

$$\lambda = \theta - \theta_0$$

where ε is the axial tilt of planet, λ is ecliptic latitude of sun, δ is the declination of sun, α is the right ascension of sun, e is the eccentricity of planet's orbit.

Simulating from the above equations gives us





Kirkwood Gaps

Kirkwood gaps are intriguing voids within the asteroid belt, a region located between the orbits of Mars and Jupiter. Discovered by astronomer Daniel Kirkwood in the mid-19th century, these gaps are regions where the density of asteroids is notably lower compared to the surrounding areas. The gaps are caused by a resonant interaction between the asteroids and Jupiter's gravitational influence. As asteroids orbit the Sun, Jupiter's powerful gravitational pull perturbs their orbits, creating specific resonances where the asteroid's orbital period is in a simple integer ratio with Jupiter's orbital period. Due to these resonances, asteroids experience gravitational interactions that either eject them from the gap or push them into different orbital configurations, resulting in the observed gaps.

The most prominent of these gaps, known as the "Kirkwood gaps," are found at specific orbital distances, such as 2.06, 2.5, and 2.82 astronomical units (AU) from the Sun. These gaps indicate regions where the orbital dynamics are less stable, causing asteroids to migrate away from these orbital resonances over time. The discovery and study of Kirkwood gaps have significantly contributed to our understanding of the complex gravitational interactions in the solar system and the long-term evolution of asteroid orbits.

We attempted to recreate these kirkwood gaps using a simplistic setup. We took 10,000 asteroids orbiting the sun in a uniform radial distribution throughout, introduced Sun's and Jupiter's gravity and tried to simulate the system for 1 million years. Since the program was very big and memory intensive however, we needed to optimise it heavily in C++. We ended up with partial results as follows