Krittika Summer Projects 2022

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Ajinkya Naik[1], Nayan Telrandhe[2], Taisiia Karasova[3], Rehmat Singh Chawla[4], and Yuvraj Sharma[5]

[1]MES Abasaheb Garware College, Pune, 411004, India

[2]University of Hyderabad, Hyderabad, 500046, India

[3]University of Groningen, Groningen, 9747 AE, The Netherlands

[4]Indian Institute of Technology Bombay, Mumbai, 400076, India

[5]Indian Institute of Science Education and Research, Bhopal, 462066, India

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Abstract

Gravitational Waves are ripples in spacetime, which were first predicted by Albert Einstein in 1916 using the General Theory of Relativity and observed experimentally in 2015 by the Laser Interferometer Gravitational wave Observatory (LIGO). The modeling of gravitational wave signals detected by LIGO requires solving the Einstein Field Equations to theoretically generate the gravitational waveforms. However, due to their highly non-linear and complicated nature we use certain approximation methods. In this project, we employed the Quadrupole Approximation and Post Newtonian Expansions to generate gravitational waveforms of Compact Binary Coalescence (CBC) for varying source parameters. A delay in the coalescence time was observed for Post Newtonian (PN) Waveform as compared to the Newtonian waveform because of the PN correction terms. The delay was found to vary with the masses of the binary components and choice of initial gravitational wave frequency. Matched Filtering of the GW150914 strain data was done with the generated waveforms as templates. A higher Signal to Noise Ratio (SNR) was obtained for the PN Waveform compared to the Newtonian waveform, which suggests its effectiveness in finding gravitational wave signals.

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I Theory

1. GR explanation of gravitational waves

1.1 Brief introduction to the General Relativity

General relativity stands as a well-tested description of Gravitation from more than 100 years. General relativity generalises Special Relativity beyond vacuum conditions i.e. in presence of heavy masses, radiation etc and describes Gravity as a geometric property of space-time.
The fundamental quantity called metric $g_{\mu\nu}$ contains all the information about the geometry of spacetime. The metric is a function of position in spacetime

$$g_{\mu\nu} = g_{\mu\nu}(x^{\alpha}) \tag{1.1}$$

where $x^{\alpha} = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ is a 4-dimensional spacetime coordinate.

In flat spacetime for a line element ds , $ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu}$ where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

$\eta_{\mu\nu}$ is called Minkowiski Metric and $g_{\mu\nu} = \eta_{\mu\nu}$ in flat spacetime.

The metric is a tensor quantity and the components of the metric are determined by Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.3}$$

where,

$T_{\mu\nu}$ is Energy Momentum Tensor - the tensor describes flux and density of energy and momentum in spacetime,

$R_{\mu\nu}$ is Ricci Tensor and R is Ricci scalar,

$R = g^{\mu\nu}R_{\mu\nu}$, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ where, $R^{\beta}{}_{\mu\alpha\nu}$ is called Reimann Tensor - it is a geometric measure of curvature of spacetime,

Λ is the Einstein's Cosmological constant, c and G being the speed of light and Gravitational constant respectively.

Very few analytic solutions are known for such an equation. Some known analytic solutions exists for Einstein's Field Equations which are linearised (1.3.1) around flat spacetime. The solutions in the Newtonian limit should match the dynamics of the interacting bodies as predicted by Newton's laws of Gravitation. This amounts to the requirement that the gravitational field be weak, static (no time derivatives) and the test particles be moving slowly. In a less restrictive situation in which field can vary with time and with no restrictions on the motion of test particles new phenomenon are observed. Gravitational radiation/waves being one among them. It is observed when field is allowed to vary with time.

1.2 Gravitational waves from Newtonian Potential

The gravitational field potential in Newtonian Gravitational formulation is given by

$$\nabla^2\phi(x,t) = 4\pi G\rho \tag{1.4}$$

where, ρ is the mass density of the source of the field. The solution of this field is given by

$$\phi_{\mathrm{N}}(\mathbf{x},t) = -G\int \rho(\mathbf{y},t)r^{-1}d^3y, r \equiv |\mathbf{x}-\mathbf{y}| \tag{1.5}$$

ϕ_{N} denotes the Newtonian field potential. It can be observed that the above field potential is instantaneous i.e. the change in the potential due to change in the mass distribution is instantaneous. This violates Special Relativity, as no information should be able to propagate faster than the speed of light c.
Thus we introduce a delay (retardation),so the change in ρ at y is felt at x after time $\frac{|x-y|}{c}$. The new field potential ϕ_R is called the relativistic field potential.

$$\phi_R(\mathbf{x},t) = -G\int \rho(\mathbf{y},t-r/c)r^{-1}d^3y, r \equiv |\mathbf{x}-\mathbf{y}| \tag{1.6}$$

It can be shown that ϕ_R satisfies scalar wave equation.

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 4\pi G\rho \tag{1.7}$$

The insertion of the retardation term leads to Gravitational Waves. The spatial gradient of ϕ_R is given by

$$\nabla\phi_R = G\int\left(\frac{\rho}{r} - \frac{1}{c}\frac{\partial\rho}{\partial t}\right)\frac{\mathbf{x}-\mathbf{y}}{r^2}d^3y \tag{1.8}$$

If we consider ρ to be non zero only in a region of radius R around the origin. Then far away from the source $|x| >> R \Rightarrow r \approx x$, then the first term is negligible compared to the second, hence

$$\hat{x}\cdot\nabla\phi_R \approx -\frac{1}{c}\frac{\partial\phi_R}{c\partial t} \tag{1.9}$$

Thus far away from the source at length scale λ, changes in ϕ_R ($\hat{x}\cdot\nabla\phi_R \sim \phi_R/\lambda$) is c times typical time scale P on which ϕ_R changes.
So a change along x is compensated for by a change in a time larger by a factor of c, which is characteristic of a wave travelling at speed c. Thus the spatial variance of relativistic potential ϕ_R is more sensitive to $\partial\rho/\partial t$ than the distance $|x|$ in contrast to Newtonian potential ϕ_N which depends only on $|x|$.
These wave like phenomenon are the Gravitational Waves.

1.3 Relativistic wave equation

1.3.1 Linearization of Metric

In an approach to the same problem using General Relativity i.e. when the field is not restricted to static situations and is allowed to vary with time, Gravitational waves are observed. For a weak gravitational field the metric can be decomposed into a flat Minkowski Metric plus a small perturbation.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| << 1 \tag{1.10}$$

where, $\eta_{\mu\nu} = diag(-1,+1,+1,+1)$

Thus, a weak gravitational field differs only slightly from flat spacetime. The quantities $h_{\mu\nu}$ are perturbations or deviations of the metric away from flat spacetime. The assumption that $h_{\mu\nu}$ is small allows us to ignore anything that is higher than first order in this quantity, we also ignore the product of the quantity with it's derivatives ($h_{\mu\nu}...\partial h_{\mu\nu}...$) and product of it's derivatives ($\partial h_{\mu\nu}...\partial h_{\mu\nu}...$) , Thus to the first order

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{1.11}$$

where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$

In weak gravitational fields one typically raises and lowers indices with the background Minkowski metric $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ and not with $g_{\mu\nu}$ and $g^{\mu\nu}$ as the corrections would be of higher order in perturbation.

1.3.2 Linearization of Einstein's Field Equations

To the first order, The **Christoffel symbols** are given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right) = \frac{1}{2} \eta^{\rho\lambda} \left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right) \tag{1.12}$$

As we restrict to the first order, the contribution to the **Reimann Tensor** will be due to the derivatives of Γs and not the Γ^2 terms, giving

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho} \\ &= \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}\right). \end{aligned} \tag{1.13}$$

Contracting over μ and ρ the **Ricci Tensor**

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^{\sigma}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}_{\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) \tag{1.14}$$

where, $h = \eta^{\mu\nu} h_{\mu\nu} = h^{\mu}_{\mu}$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{\partial^2}{\partial t^2} + \nabla^2$

which on further contraction gives **Ricci scalar**

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \tag{1.15}$$

Using expressions for Ricci Tensor and Ricci scalar the left hand side of the Einstein's Field equation

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \\ &= \frac{1}{2}\Big(\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \eta_{\mu\nu}\Box h\Big) \end{aligned} \tag{1.16}$$

here, $G_{\mu\nu}$ is called as the **Einstein Tensor**.
$G_{\mu\nu}$ can be rewritten as

$$G_{\mu\nu} = \frac{1}{2}(\partial^\sigma\partial_\nu h_{\sigma\mu} + \partial^\sigma\partial_\mu h_{\sigma\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \eta_{\mu\nu}\Box h) \tag{1.17}$$

we define $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}$, which on substitution

$$G_{\mu\nu} = \frac{1}{2}(\partial^\sigma\partial_\nu \bar{h}_{\sigma\mu} + \partial^\sigma\partial_\mu \bar{h}_{\sigma\nu} - \eta_{\mu\nu}\partial^\alpha\partial^\beta \bar{h}_{\alpha\beta} - \Box\bar{h}_{\mu\nu}) \tag{1.18}$$

1.3.3 Equations in Lorenz Gauge

A change of coordinates can be done where the first three terms can be set to 0. This coordinate system is called 'Lorenz Gauge'.

In 'Lorenz Gauge' , $\partial_\beta \bar{h}^{\alpha\beta} = 0$ called as the 'Lorenz condition'.

This condition simplifies $G_{\mu\nu}$ such that

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} \tag{1.19}$$

In this gauge the linearized Einstein's field equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ simplifies further to

$$\begin{aligned} -\frac{1}{2}\Box\bar{h}_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h &= -16\pi G T_{\mu\nu} \end{aligned} \tag{1.20}$$

The vacuum equation (when $T_{\mu\nu} = 0$) is

$$\Box\bar{h}_{\mu\nu} = 0 \tag{1.21}$$

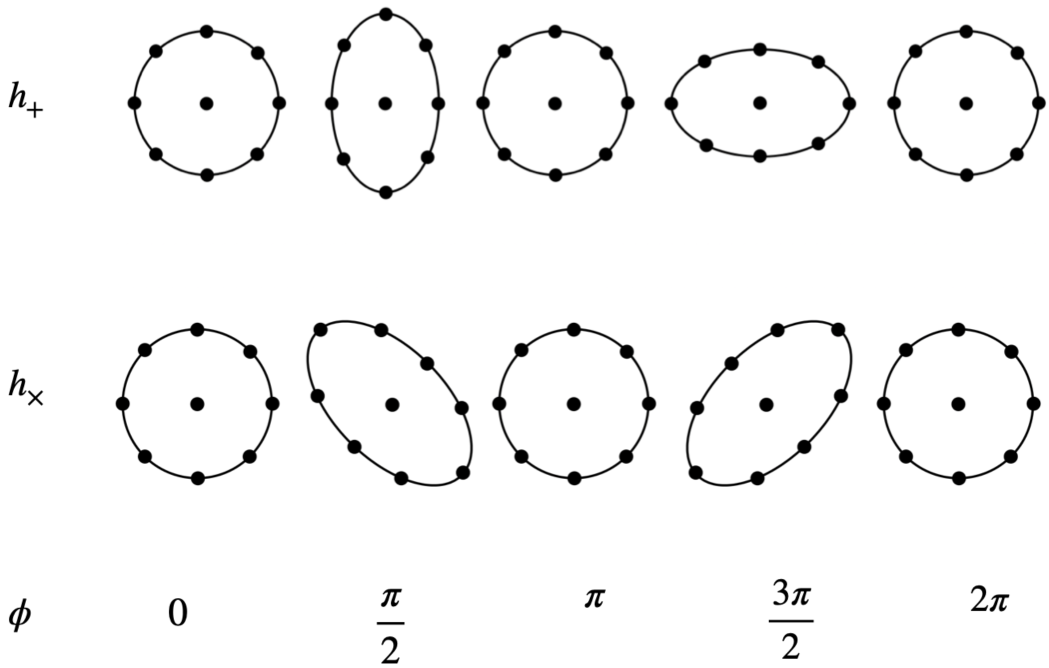
The obtained equation is a relativistic wave equation. It can be written as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{h}_{\mu\nu} = 0 \tag{1.22}$$

whose solution is a waveform travelling at speed of light 'c'. The above equation determines the evolution of a disturbance in the gravitational field in vacuum in the harmonic gauge.

Note

The linear metric perturbation $h_{\mu\nu}$ has 16 components. There exists a choice of coordinates called 'Traceless Transverse Gauge' where the number of independent components reduce to two components. These independent components correspond to two Polarization of Gravitational waves - 'Plus Polarization' ($+$) and 'Cross Polarization' ($\times$). Thus all polarizations in this Linearized Gravity formulation are combination of Plus and Cross polarizations with different amplitude and phases. A Plus polarized gravitational wave causes a ring of free particles to stretch and compress along the directions of a Plus sign and a Cross polarized gravitational wave causes same along the cross sign.



1.4 Quadrupole approximation

Solving relativistic wave equation, one get the approximation describing gravitational wave strain

Proposition 1.4.1 Gravitational wave strain components are calculated in the following way

$$h_{ij} = \frac{2G}{c^4 d} \cdot \frac{d^2 Q_{ij}}{dt^2} \tag{1.23}$$

Here G is the gravitational constant, $c-$ speed of light in vacuum, $d-$ distance from the system to the observer. $Q_{ij}-$ system's quadrupole tensor defined below

Strain is then calculated in the following way

$$|h|^2 = \sum_{i,j=1}^{3} h_{ij} h_{ij} \tag{1.24}$$

Definition 1.4.1 — Quadrupole moment of a system's mass distribution. Given that density distribution of a system is $\rho(\mathbf{x})$ (in Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$) quadrupole tensor is defined as

$$Q_{ij} = \int d^3x \rho(\mathbf{x}) \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right) \tag{1.25}$$

Where r is the radial distance from the origin, δ_{ij} is the Kronecker-delta symbol. Integration occurs throughout the whole space

Binary systems play a crucial role in gravitational radiation studies and were the focus of this project. Therefore, it is vitally important to make calculations for this type of systems.
Further we will use several definitions connected to binary system's components' masses m_1 and m_2

Definition 1.4.2

$$M = m_1 + m_2 \tag{1.26}$$

Definition 1.4.3 — Reduced mass.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{1.27}$$

Definition 1.4.4 — Chirp mass.

$$\mathscr{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \tag{1.28}$$

■ **Example 1.1 — Quadrupole moment of a binary system.** Let ωt be the angle between the line connecting objects of a binary system and $x-$axis. Then, the quadrupole tensor components of the system are

$$Q_{ij}(t) = \frac{1}{2} \mu r^2 I_{ij} \tag{1.29}$$

where $I_{xx} = \frac{1}{3} + \cos 2\omega t$, $I_{yy} = \frac{1}{3} - \cos 2\omega t$, $I_{xy} = I_{yx} = \sin 2\omega t$, $I_{zz} = -\frac{2}{3}$, $I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$
■

A very important consequence of this expression is strain-dependence evolution, which could be received from (1.24) and (1.29). GR expression for it is given below. From (1.29) we get that frequency f of gravitational waves emitted by the system is equal to twice the orbital frequency of the system $(\omega/2\pi)$

Proposition 1.4.2 Strain evolution of binary system's gravitational waves

$$h(t) = h_0 \cos(2\pi f t + \pi \dot{f} t^2 + \phi_0) \tag{1.30}$$

where ϕ_0 is the starting phase
Scaling amplitude is

$$h_0 = 4 \frac{G}{c^2} \frac{\mathscr{M}}{d} \left(\frac{G}{c^3} \pi f \mathscr{M} \right)^{2/3} \tag{1.31}$$

Graph of amplitude-time dependence is represented at the Figure 1.1

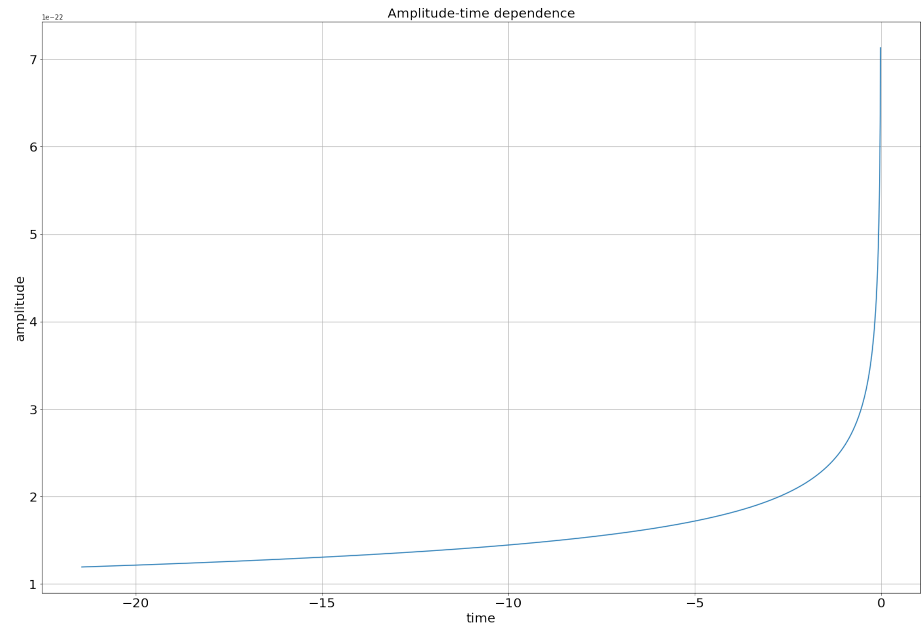


Figure 1.1: Time evolution of a binary system's (with masses $10M_\odot$ and $20M_\odot$) gravitational waves amplitude. Distance to the observer $D = 300Mpc$

1.5 Frequency-time dependence

Given that the frequency f of gravitational waves emitted by the system is equal to twice the orbital frequency of the system $(\omega/2\pi)$, from the fact that power of gravitational radiation = rate of system's energy $\left(E_{system} = -\frac{GM\mu}{r}\right)$ change and , after integration we get formula for frequency evolution of gravitational waves emitted by the binary system

Proposition 1.5.1 Frequency-time dependence for binary system's gravitational emission

$$f^{-8/3} = \frac{(8\pi)^{8/3}}{5}\left(\frac{G\mathscr{M}}{c^3}\right)^{5/3} t \tag{1.32}$$

where t is time left to merger of two objects

Graph of frequency-time dependence is represented at the Figure 1.2

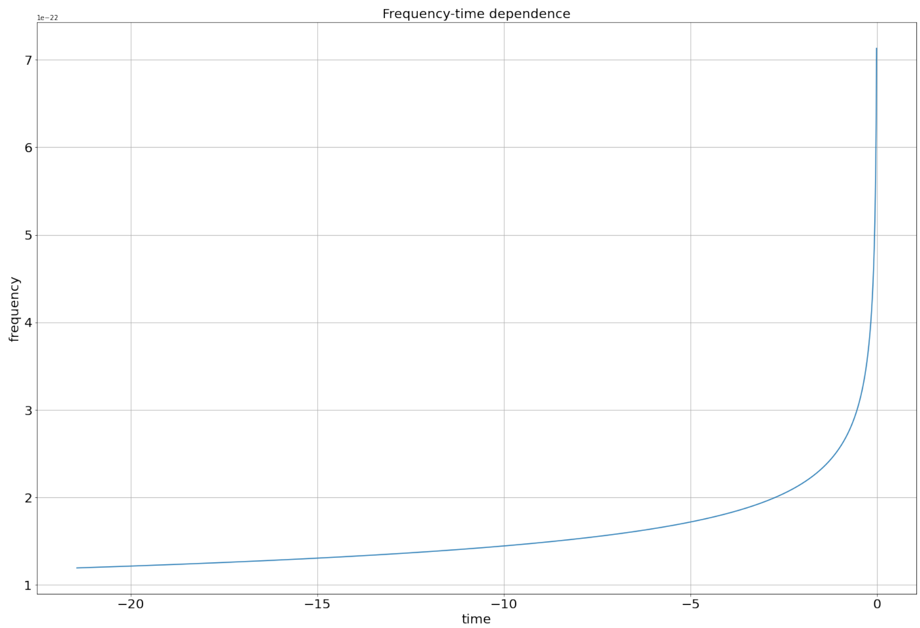


Figure 1.2: Time evolution of a binary system's (with masses $10M_{\odot}$ and $20M_{\odot}$) gravitational waves frequency. Distance to the observer $D = 300Mpc$

1.6 Power of gravitational radiation

Using expressions derived above, one can calculate power of gravitational radiation of the system

$$P = \frac{c^3}{16\pi G}\int |\dot{h}|^2 \tag{1.33}$$

Here integration happens over a sphere of radius d. Substituting (1.24) and (1.23) to (1.33), one get expression for power of gravitational radiation through quadrupole tensor

Proposition 1.6.1 Power of gravitational radiation

$$P = \frac{1}{5} \cdot \frac{G}{c^6} \sum_{i,j=1}^{3} \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3} \tag{1.34}$$

As already said, the focus of the project was gravitational radiation of binaries. Therefore, it would be useful to calculate power of gravitational radiation in this particular case
Using (1.29), (1.34)

■ **Example 1.2 — Gravitational radiation of binary system.**

$$P = \frac{32}{5} \frac{G\mu^2 r^4 \omega^6}{c^5} \tag{1.35}$$

Using Kepler's third law $\left(r^3 = GM/\omega^2\right)$, one can derive

$$P = \frac{32}{5} \frac{G^4 M^5}{c^5 r^5} \tag{1.36}$$

■

1.7 Limits of binary system model acceptability and resulting waveform

All formulas derived above for binary system gravitational radiation are applicable until merger. Consequently, it would be very useful to understand at which frequency and strain we can assume that the system is about to merge. This happens when the distance between the two objects is equal to the radius R_{ISCO} of the innermost stable circular orbit.

Proposition 1.7.1 Radius of the innermost stable orbit of a circular binary system

$$R_{ISCO} = \frac{6GM}{c^2} \tag{1.37}$$

Given that the frequency f of gravitational waves emitted by the system is equal to twice the orbital frequency of the system $(\omega/2\pi)$ and Kepler's third law, one can derive the frequency at which we assume merger happens

$$f_{ISCO} = \sqrt{\frac{GM}{\pi^2 R_{ISCO}^3}}$$

Proposition 1.7.2 Gravitational waves frequency at merger of the binary system

$$f_{ISCO} = \frac{c^3}{\pi GM} \sqrt{\frac{1}{216}} \tag{1.38}$$

Taking into account everything derived above, one receives the following waveform (Figure 1.3)

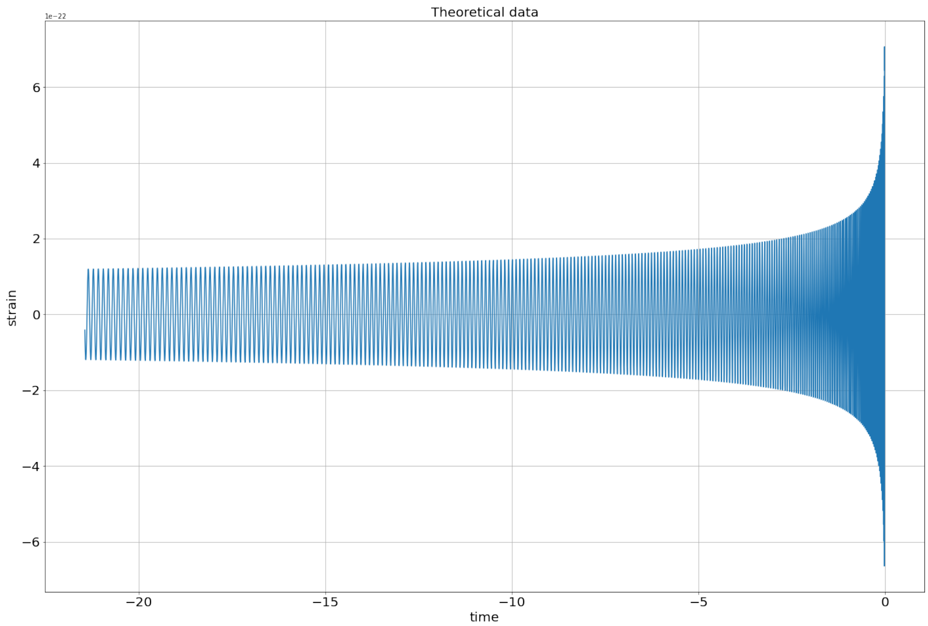


Figure 1.3: Binary system's (with masses $10M_{\odot}$ and $20M_{\odot}$) gravitational waves waveform. Distance to the observer $D = 300Mpc$

2. Post Newtonian (PN) Theory

2.1 Introduction

The Newtonian Theory of Gravity conveniently suffices for most of astrophysical calculations. The General Theory of Relativity on the other hand, also bears a Newtonian Limit in which it reduces to Newtonian Gravity for non-relativistic scenarios. In section 1.2, we showed how it is possible to demonstrate the existence of gravitational waves just by introducing a time delay in the Newtonian Potential (ϕ). The resulting Quadrupole Moment Formalism gives us a good first hand approximation and differs from the results given by General Relativity only in some aspects. However, they are based on the assumption that the motion of the source and the spacetime curvature are independent. Our astrophysical systems of interest for gravitational wave detection are bound by gravitational forces which renders this assumption invalid. Thus, for moderately relativistic systems, we need a model that takes into account the effect of spacetime curvature on the velocity of source and consequently on the gravitational radiation.

The Post Newtonian Expansion theory is a model which is used to find approximate solutions to the Einstein Field Equations using expansions in terms of a parameter (ε). Post Newtonian Expansions were first used by Albert Einstein in calculating the precession of the perihelion of Mercury's orbit, which served as one of the first tests of General Relativity.

2.1.1 Newtonian Limit of General Relativity

In non-relativistic conditions such as slow velocities or weak and static gravitational fields, General Relativity and its equations of motion reduces to the Newtonian case. If we assume that the source is moving slowly then the time component of its 4-velocity will dominate the spatial component i.e. the source will have a time-like velocity :

$$\frac{dx^i}{d\tau} << \frac{dt}{d\tau} \tag{2.1}$$

where x^i is the i^{th} coordinate in some reference frame and τ is the proper time. The time-like geodesic equation is given by :

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

Due to result (2.1), the geodesic equation becomes :

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{2.2}$$

We use the relation between the Christoffel symbols and metric tensor to evaluate Γ^μ_{00} as :

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda 0}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right)$$

Since, we have assumed static gravitational fields for the Newtonian Limit, the first two terms inside the bracket will vanish and only the final term is retained. Thus,

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{00}}{\partial x^\lambda}\right) \tag{2.3}$$

We saw that for a weak gravitational field, the general metric tensor $g_{\mu\nu}$ can be expressed in terms of the Minkowski Metric $\eta_{\mu\nu}$ plus some small perturbation $h_{\mu\nu}$ i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The matrix $g_{\mu\nu}$ is invertible, therefore we can write :

$$g^{\mu\nu}g_{\nu\alpha} = \delta^\mu_\alpha = I$$

where I is the identity matrix. If the perturbation $h_{\mu\nu}$ is really small then one can say

$$(\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\mu\nu} + h_{\mu\nu}) = I = g^{\mu\nu}g_{\nu\alpha} \quad \text{, because } h_{\mu\nu}h^{\mu\nu} \approx 0$$

which implies that :

$$g^{\mu\nu} = (\eta^{\mu\nu} - h^{\mu\nu})$$

and also :

$$g_{\mu\nu} = (\eta_{\mu\nu} + h_{\mu\nu})$$

For $\mu = 0, \nu = 0$ we get

$$g_{00} = \eta_{00} + h_{00} = 1 + h_{00} \quad \text{and consequently} \tag{2.4}$$

$$\frac{\partial g_{00}}{\partial x^\lambda} = \frac{\partial h_{00}}{\partial x^\lambda} \tag{2.5}$$

The Christoffel Symbol in eqn (2.3) can then be written as :

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\lambda}\frac{\partial h_{00}}{\partial x^{\lambda}} \tag{2.6}$$

and the geodesic equation (2.2) becomes :

$$\frac{d^2x^{\mu}}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\left(\frac{\partial h_{00}}{\partial x^{\lambda}}\right)\left(\frac{dt}{d\tau}\right)^2 \tag{2.7}$$

Now, as the gravitational field is static, $\frac{\partial h_{00}}{\partial x^0} = 0$ and for $x^{\mu} = x^0 = t$, eqn (2.7) becomes :

$$\frac{d^2t}{d\tau^2} = 0 \tag{2.8}$$

For the spatial components ($\mu \neq 0$) one can write :

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\left(\frac{\partial h_{00}}{\partial x^{\lambda}}\right)\left(\frac{dt}{d\tau}\right)^2 \quad \text{where i = 1,2,3}$$

If we write $\frac{d^2x^i}{d\tau^2}$ as $\frac{d}{d\tau}\left(\frac{dx^i}{d\tau}\right)$, then by applying the product rule it simplifies into :

$$\frac{d^2x^i}{d\tau^2} = \left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x^i}{dt^2}\right) + \frac{dx^i}{dt}\left(\frac{d^2t}{d\tau^2}\right)$$

From result (2.8), the second term will reduce to zero. Therefore

$$\frac{d^2x^i}{d\tau^2} = \left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x^i}{dt^2}\right)$$

Multiplying by $\frac{d\tau}{cdt^2}$ on both sides,

$$\frac{d^2x^i}{c^2dt^2} = -\frac{1}{2}\left(\frac{\partial h_{00}}{\partial x^i}\right) \tag{2.9}$$

If we assume that $h_{00} = \frac{2\phi}{c^2}$, where ϕ is the Newtonian gravitational potential then (2.9) reduces to :

$$\frac{d^2x^i}{dt^2} = -\left(\frac{\partial \phi}{\partial x^i}\right) \tag{2.10}$$

In other words, as $\frac{d^2x^i}{dt^2}$ is just the spatial acceleration and $\frac{\partial}{\partial x^i}$ is the spatial gradient, equation (2.10) can be written as :

$$\mathbf{g} = -\nabla\phi \quad \text{where g is acceleration due to gravity} \tag{2.11}$$

which is nothing but an alternate statement of Newton's Law of Gravitation. Thus, under appropriate conditions and assumptions, the General Theory of Relativity reduces to Newton's Law of Gravitation. Furthermore, we assumed that $h_{00} = \frac{2\phi}{c^2}$, thus from equation 2.4:

$$g_{00} = 1 + \frac{2\phi}{c^2} \tag{2.12}$$

2.2 The Small Parameter and Constraints of PN Theory

The Post Newtonian Theory is an effective way of obtaining an approximate solution to the Einstein Field Equations. In a way, this theory quantifies the deviation from a completely Newtonian source towards moderately relativistic sources and depicts the non-linearity of General Relativity. This is achieved by expanding the approximate solutions in terms of n- orders of a parameter ε which is a ratio of the source velocity v to the speed of light c : (v/c). Such terms will be called as the Post-Newtonian corrections of n^{th} order or $\frac{n}{2}$ PN Terms. It is also observed that:

$$\varepsilon \sim \sqrt{\frac{R_s}{d}} \sim \frac{v}{c}$$

Where $R_s = 2GM/c^2$ is the equivalent of Schwarzschild Radius of the source and d is the size of the system (for e.g. orbital radius in case of a binary system).

Thus ε also characterizes the compactness of source. This Post Newtonian Expansion is applicable in the domain of moderately relativistic and weakly self-gravitating sources ($\varepsilon << 1$) in the Near Field Region only, for which the parameter ε has a small value and retardation effects due to the gravitational waves are negligible. It is assumed that the Stress Energy Momentum Tensor for the source $T^{\mu\nu}$ has a spatially compact support i.e. the source can be enclosed in a time-like tube of world lines for which $r \leq d$. We also assume that the matter inside the source is smooth and continuous which implies that $T^{\mu\nu}(t,x)$ is infinitely differentiable over the entire space-time. In addition, we impose the condition that the source be weakly stressed i.e.

$$\frac{T^{ij}}{T^{00}} = O(\varepsilon^2)$$

For a fluid, the first entry of the Stress Energy Momentum Tensor (T^{00}) corresponds to the energy density ρ. If p is the pressure in fluid then :

$$\frac{p}{\rho} = O(\varepsilon^2)$$

In the limit of $\varepsilon \to 0$, the Post Newtonian Approximation reduces to the Newtonian theory for non-relativistic sources. In case of a Compact Binary Coalescence, Post Newtonian Approximation is effective throughout the inspiral phase. Since, at the merger phase, relativistic effects take precedence and ε is no longer small. In this regime, other models such as Numerical Relativity and Effective One-Body Simulation are required.

2.3 Post Newtonian Expansion of Einstein Equations

Having introduced the parameter ε, we then expand the metric tensor g_{ij} and the Stress Energy Momentum Tensor T^{ij} in powers of ε. The metric tensor is expanded as follows :

$$\begin{aligned} g_{00} &= -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + ... \\ g_{0i} &= {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + ... \\ g_{ij} &= \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + ... \end{aligned} \tag{2.13}$$

The Energy Momentum Tensor is expanded as :

$$\begin{aligned} T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + ... \\ T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + ... \\ T^{ij} &= {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + ... \end{aligned} \tag{2.14}$$

where ${}^{(n)}g_{\mu\nu}$ and ${}^{(n)}T^{\mu\nu}$ denote terms of order ε^n in the expansion.

These expansions are then substituted in the Einstein Field Equations and terms of the same order in ε are equated. In Post Newtonian Expansions, we assume an almost non-relativistic motion of source. Due to this, the d'Alembertian operator or square operator $\Box$ applied to the metric to the lowest order reduces to ordinary Laplacian operator ∇^2 i.e.

$$-\frac{1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2 = [1 + O(\varepsilon^2)]\nabla^2 \tag{2.15}$$

This implies that retardation effects in the gravitational potential ϕ are small corrections. In lowest order, the solution can be approximated as instantaneous potentials without any retardation effects. Therefore, we are computing some quantity $F(t-r/c)$ which is an intrinsic function of the retarded time $t-r/c$, in terms of expansions for small retardation :

$$F(t-r/c) = F(t) - \frac{r}{c}\dot{F}(t) + \frac{r^2}{2c^2}\ddot{F}(t) + ... \tag{2.16}$$

Each derivative of F with respect to time has some factor of ω, where ω is angular frequency of the gravitational radiation emitted.

R Since $\omega/c = 1/\lambda\!\!\!^{-}$ (where $\lambda\!\!\!^{-}$ is the reduced wavelength),it can be seen that equation 2.4 is actually an expansion in powers of $r/\lambda\!\!\!^{-}$.Thus, the expansion will break down in the radiation zone when $r >> \lambda\!\!\!^{-}$. This is the reason as to why the Post Newtonian Expansion is valid in the near zone where $r << \lambda\!\!\!^{-}$.

2.4 1 PN Corrections

We employ a gauge condition (co-oordinate choice) known as the De-Donder gauge (or Harmonic Gauge) :

Definition 2.4.1 The De-Donder gauge condition is defined as : $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$. The resulting coordinates are called as Harmonic coordinates.

An appropriate choice of gauge condition can drastically simplify our calculations and expressions. By substituting the expansions (2.12) and (2.13) in the Einstein equations and using the gauge condition (2.4.1) expanded to the required order, for ${}^{(2)}g_{00}$ we get :

$$\nabla^2[{}^{(2)}g_{00}] = -\frac{8\pi G}{c^4}\,{}^{(0)}T^{00} \tag{2.17}$$

and for 1 PN corrections to the metric :

$$\nabla^2[{}^{(2)}g_{ij}] = -\frac{8\pi G}{c^4}\delta^i_j\,{}^{(0)}T^{00} \tag{2.18}$$

$$\nabla^2[{}^{(3)}g_{0i}] = \frac{16\pi G}{c^4}\,{}^{(1)}T^{0i} \tag{2.19}$$

$$\nabla^2[{}^{(4)}g_{00}] = \partial_0^2[{}^{(2)}g_{00}] + {}^{(2)}g_{ij}\partial_i\partial_j[{}^{(2)}g_{00}] - \partial_i[{}^{(2)}g_{00}]\partial_j[{}^{(2)}g_{00}] \tag{2.20}$$
$$-\frac{8\pi G}{c^4}[{}^{(2)}T^{00} + {}^{(2)}T^{ii} - 2\;{}^{(2)}g_{00}\;{}^{(0)}T^{00}]$$

Under the boundary condition that the metric vanishes at spatial infinity ($r \to \infty$), the solution for equation (2.17) is :

$$ {}^{(2)}g_{00} = -2\phi \tag{2.21}$$

where,

$$\phi(t,x) = -\frac{G}{c^4}\int d^3x'\frac{{}^{(0)}T^{00}(t,x')}{|x-x'|} \quad \text{is the negative Newtonian potential.} \tag{2.22}$$

R It is observed that this solution (2.21) for the expanded metric ${}^{(2)}g_{00}$ is also apparent from the result (2.12) obtained in evaluating the Newtonian limit of GR.

Similarly, the 1 PN equations for other metric components can be solved to give :

$$ {}^{(2)}g_{ij} = -2\phi\delta^i_j \tag{2.23}$$

$$ {}^{(3)}g_{0i} = \zeta_i \quad \text{for,} \tag{2.24}$$

$$\zeta_i(t,x) = -\frac{4G}{c^4}\int d^3x'\frac{{}^{(1)}T^{0i}(t,x')}{|x-x'|} \tag{2.25}$$

Finally, to solve equation (2.20) we substitute the solutions (2.21) and (2.23) into the right hand side. Writing ${}^{(4)}g_{00}$ as :

$$ {}^{(4)}g_{00} = -2(\phi^2+\psi) \tag{2.26}$$

where ψ is some potential.
Using the identity :

$$\partial_i \partial_j \phi = \frac{1}{2}\nabla^2(\phi^2) - \phi\nabla^2\phi \tag{2.27}$$

equation (2.20) then becomes :

$$\nabla^2\psi = \partial_0^2 + \frac{4\pi G}{c^4}[^{(2)}T^{00} +^{(2)} T^{ii}] \tag{2.28}$$

Similar to equation (2.17) with the boundary condition that ψ goes to zero at infinity, the above equation has a solution :

$$\psi(t,x) = -\int \frac{d^3x'}{|x-x'|}\left(\frac{1}{4\pi}\partial_0^2\phi + \frac{G}{c^4}\left[^{(2)}T^{00}(t,x') +^{(2)} T^{ii}(t,x')\right]\right) \tag{2.29}$$

As discussed before, since we are computing the PN terms at a lower order of n=2, the solutions ϕ, ζ and ψ are actually instantaneous potentials. They depend on the value of Energy Momentum tensor at time t rather than at the delayed time $t - r/c$. However, it is possible to express the solution in terms of retarded potentials evaluated at the delayed time. The 1 PN Corrections for g_{00} can be written as :

$$\Box V = -4\pi G\sigma \tag{2.30}$$

The solution $V(t,x)$ is then expressed as a retarded integral :

$$V(t,x) = G\int d^3x' \frac{1}{|x-x'|}\sigma\left(t - \frac{|x-x'|}{c}, x'\right) \tag{2.31}$$

$V(t,x)$ can be given in terms of instantaneous potentials by expanding the $\sigma\left(t - \frac{|x-x'|}{c}, x'\right)$ for small retardation effects till the 1 PN Order. Similarly, for g_{0i} and g_{ij} we have :

$$V_i(t,x) = G\int d^3x' \frac{1}{|x-x'|}\sigma_i\left(t - \frac{|x-x'|}{c}, x'\right) \tag{2.32}$$

In terms of these functions then, the 1 PN solutions are :

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + O\left(\frac{1}{c^6}\right) \tag{2.33}$$

$$g_{0i} = -\frac{4}{c^3}V_i + O\left(\frac{1}{c^5}\right) \tag{2.34}$$

$$g_{ij} = \delta^i_j\left(1 + \frac{2}{c^2}V\right) + O\left(\frac{1}{c^4}\right) \tag{2.35}$$

2.5 Radiation Reaction and 2.5 PN Terms

It is known that the emission of gravitational radiations from a source carry energy with them. According to the Law of Conservation of Energy, this energy must be derived from the source itself, which in turn should affect the motion of the source. This is known as "Radiation Reaction". The Post Newtonian corrections of higher order account for this back reaction or radiation reaction due to gravitational waves on the source. To determine the radiation reaction, we start by computing the gravitational potential inside the source. It is possible to start with the Relativistic Potential introduced in section 1.2 and expand it in powers of r/c which is the usual procedure of Post Newtonian Expansions. Starting with the Relativistic potential:

$$\phi_R(x,t) = -G\int \rho(y,t-r/c)r^{-1}d^3y, r \equiv |\mathbf{x}-\mathbf{y}| \tag{2.36}$$

We assume that x and y are of the same order and expand the mass density (ρ) in terms of r/c around $\rho(t)$.

$$\phi_R = -G\int r^{-1}\sum_{n=0}^{\infty}\left(-\frac{r}{c}\right)^n \frac{1}{n!}\frac{d^n}{dt^n}\rho(y,t)d^3y \tag{2.37}$$

This is a Near Zone expansion, which is the domain in which the Post Newtonian theory is applicable. Technically, the order of expansion n can run till infinity but we will truncate the expansion up to the 5th order ($n=5$):

$$\phi_R = -G\int r^{-1}\sum_{n=0}^{5}\left(-\frac{r}{c}\right)^n \frac{1}{n!}\frac{d^n}{dt^n}\rho(y,t)d^3y \tag{2.38}$$

A total of six terms are obtained as a result. The first term in this expansion corresponding to $n=0$ is:

$$\phi_N = -G\int \rho r^{-1}d^3y, \tag{2.39}$$

which naturally represents the Newtonian potential (ϕ_N).
The factors of r cancel in second term and we get:

$$\int \dot{\rho}d^3y = 0 \tag{2.40}$$

The third term is:

$$\phi_{PN} = -\frac{G}{2c^2}\int r\ddot{\rho}d^3y \tag{2.41}$$

It represents the first Post Newtonian or 1 PN term, which can also be obtained from (eqn 2.22).
The fourth term is independent of x and can be neglected. Next, the fifth term corresponding to $n=4$:

$$\phi_{2PN} = -\frac{G}{24c^4}\frac{d^4}{dt^4}\int \rho r^3 d^3y \tag{2.42}$$

Represents the 2 PN correction.
Finally, the sixth term for $n = 5$:

$$\phi_{2.5PN} = \frac{G}{30c^5}\left((x_i x_j Q_{ij}^{(5)} + \frac{1}{2}|x|^2 Q_{kk}^{(5)} - x_i T_i^{(5)}\right) \tag{2.43}$$

where $Q_{ij}^{(5)}$ is the fifth time derivative of the Quadrupole Tensor Q_{ij} and

$$T_i = \int \rho y_i |y|^2 d^3 y \tag{2.44}$$

Equation 2.42 gives us the 2.5 PN term. This is the term that accounts for the Radiation Reaction due to emission of gravitational waves.

2.6 The Post Newtonian Waveform

The 2 PN Waveform is generated by determining the frequency evolution $f(t)$, amplitude evolution $h(t)$ and phase evolution $\psi(t)$ as functions of time with Post Newtonian Corrections.

2.6.1 Frequency-time dependence

In order to get the frequency evolution, we can start with the Newtonian frequency relation (1.32) from section (1.5).

$$f^{-8/3} = \frac{(8\pi)^{8/3}}{5}\left(\frac{G\mathscr{M}}{c^3}\right)^{5/3}(t_c - t) \tag{2.45}$$

Here t_c is the time of coalescence. We set it to 0 and consider negative time arguments for t. This equation can be modified and written as:

$$t - t_* = \tau_0\left[1 - \left(\frac{f(t)}{f_*}\right)^{-8/3}\right] \tag{2.46}$$

where, (2.47)

$$\tau_0 = \frac{5}{256\pi} f_*^{-1}(\pi M f_*)^{-\frac{5}{3}}\nu^{-1} \tag{2.48}$$

(2.49)

$$M = \frac{Gm}{c^3} \tag{2.50}$$

$m = m_1 + m_2$ being the total mass of the system and $\nu = \mu/m$ (2.51)

t_* is any arbitrary reference time and f_* is the frequency at that time. t_* could be for example the time at which the gravitational wave signal enters into the detector. Since, the lower bound of the LIGO frequency sensitivity range is 10 Hz, f_* can be considered as 10 Hz.

We can now define the Post Newtonian corrections up to the 4th order using the following constants :

$$\tau_1 = \frac{5}{192\pi} f_*^{-1}(\pi M f_*)^{-1}\nu^{-1}\left(\frac{743}{336} + \frac{11}{4}\nu\right) \tag{2.52}$$

$$\tau_{1.5} = \frac{1}{8} f_*^{-1}(\pi M f_*)^{-\frac{2}{3}}\nu^{-1} \tag{2.53}$$

$$\tau_2 = \frac{5}{128\pi} f_*^{-1}(\pi M f_*)^{-\frac{1}{3}}\nu^{-1}\left(\frac{3058673}{1016064} + \frac{5429}{1008}\nu + \frac{617}{144}\nu^2\right) \tag{2.54}$$

The constants $\tau_1, \tau_{1.5}$ and τ_2 represent the 1 PN, 1.5 PN and 2 PN corrections respectively. In terms of these constants, the equation (2.46) receive the PN corrections as:

$$t - t_* = \tau_0\left[1 - \left(\frac{f(t)}{f_*}\right)^{-8/3}\right] + \tau_1\left[1 - \left(\frac{f(t)}{f_*}\right)^{-2}\right] - \tau_{1.5}\left[1 - \left(\frac{f(t)}{f_*}\right)^{-5/3}\right] + \tau_2\left[1 - \left(\frac{f(t)}{f_*}\right)^{-4/3}\right] \tag{2.55}$$

The "chirp" of frequency (df/dt) using this equation can be expressed as:

$$\frac{df}{dt} = \frac{3f_*}{8\tau_0}\left(\frac{f}{f_*}\right)^{11/3}\left[1 - \frac{3}{4}\frac{\tau_1}{\tau_0}\left(\frac{f}{f_*}\right)^{2/3} + \frac{5}{8}\frac{\tau_{1.5}}{\tau_0}\left(\frac{f}{f_*}\right) - \frac{1}{2}\left(\frac{\tau_2}{\tau_0} - \frac{9}{8}\left(\frac{\tau_1}{\tau_0}\right)^2\right)\left(\frac{f}{f_*}\right)^{4/3}\right] \tag{2.56}$$

By solving this differential equation numerically, we obtain the frequency evolution of gravitational waves. For $m_1 = m_2 = 35M_\odot$ the frequency domain plot is as follows :

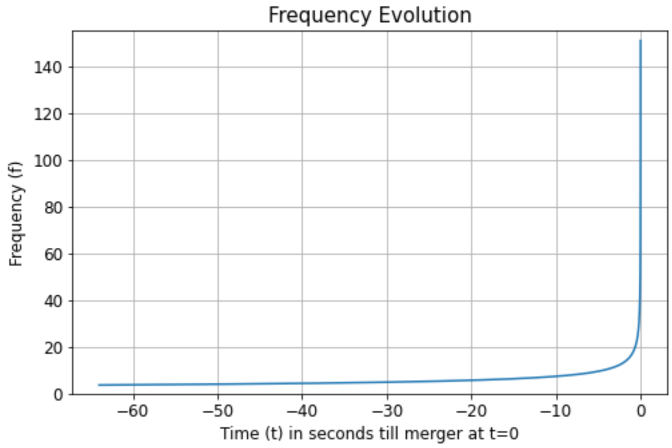


Figure 2.1: Frequency-time dependence using PN terms.

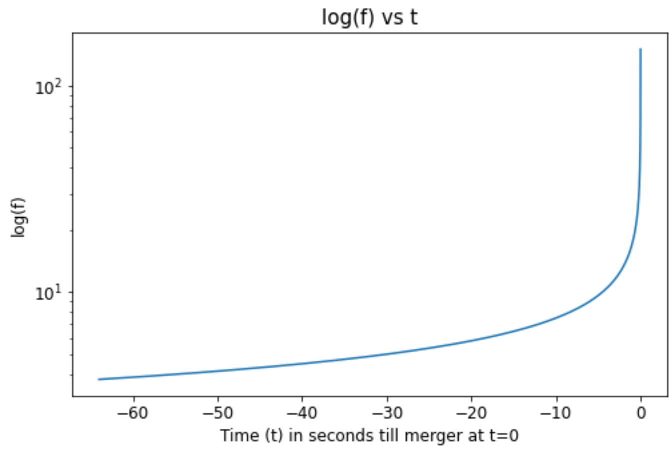


Figure 2.2: log(f) vs t using PN terms.

2.6.2 Phase Evolution and Amplitude Evolution

The accumulated phase $\psi(f(t))$ which depends on frequency $f(t)$ after receiving 2 PN corrections becomes :

$$\psi(t) = \frac{16}{5}\tau_0 f_* \left[\left(1-\left(\frac{f}{f_*}\right)^{-5/3}\right) + \frac{5}{4}\frac{\tau_1}{\tau_0}\left(1-\left(\frac{f}{f_*}\right)^{-1}\right) - \frac{25}{16}\frac{\tau_{1.5}}{\tau_0}\left(1-\left(\frac{f}{f_*}\right)^{-2/3}\right) + \frac{5}{2}\frac{\tau_2}{\tau_0}\left(1-\left(\frac{f}{f_*}\right)^{-1/3}\right)\right] \quad (2.57)$$

The amplitude evolution $h_+(t)$ is same as that of the Newtonian case. However, the amplitude phase δ_+ receives Post Newtonian corrections and is written as:

$$\delta_+(t) = 2\pi f(t_c + r/c) - \psi_0 - \frac{\pi}{4} + 2\pi f_* \left[\frac{3\tau_0}{5}\left(\frac{f}{f_*}\right)^{-5/3} + \tau_1\left(\frac{f}{f_*}\right)^{-1} - \frac{3\tau_{1.5}}{2}\left(\frac{f}{f_*}\right)^{-2/3} + 3\tau_2\left(\frac{f}{f_*}\right)^{-1/3}\right] \quad (2.58)$$

Using these terms of frequency, phase and amplitude evolution, the general wave equation can be used to generate the Post Newtonian Waveforms.

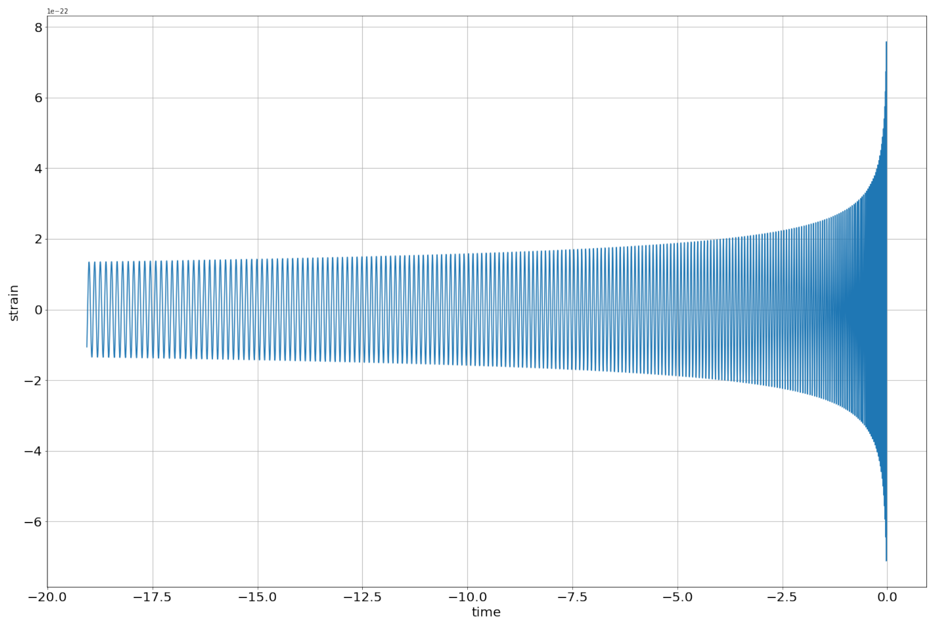
II

Results

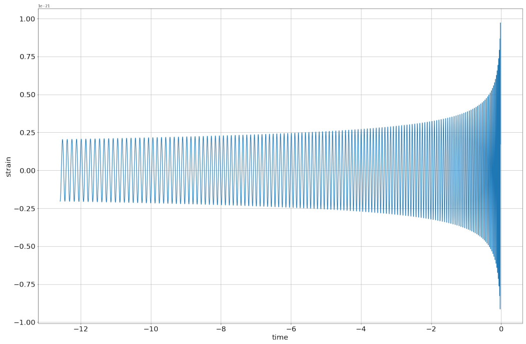
3. Waveforms

In the following chapters, wave-forms for a binary system at $D = 300\,Mpc$ are given.

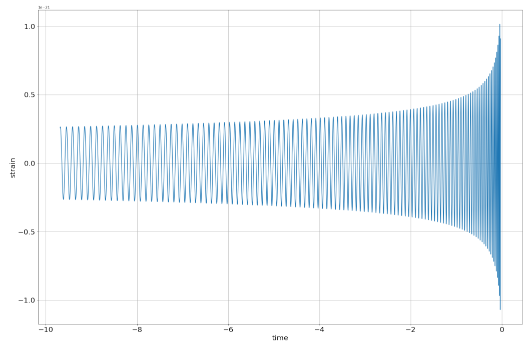
3.1 Newtonian Approximation



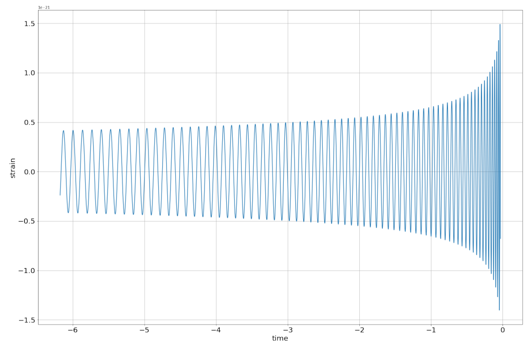
(a) Binary system's (with masses $15M_{\odot}$ and $15M_{\odot}$) gravitational waves waveform obtained using the Newtonian approximation



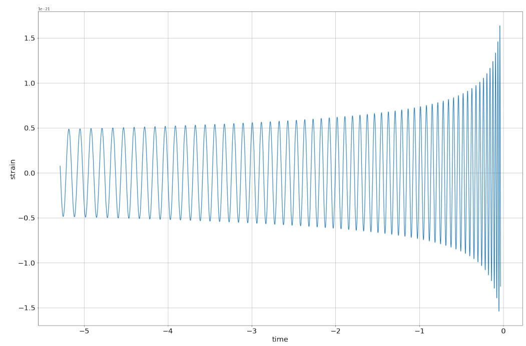
(a) Masses $15M_{\odot}$ and $25M_{\odot}$



(b) Masses $15M_{\odot}$ and $35M_{\odot}$



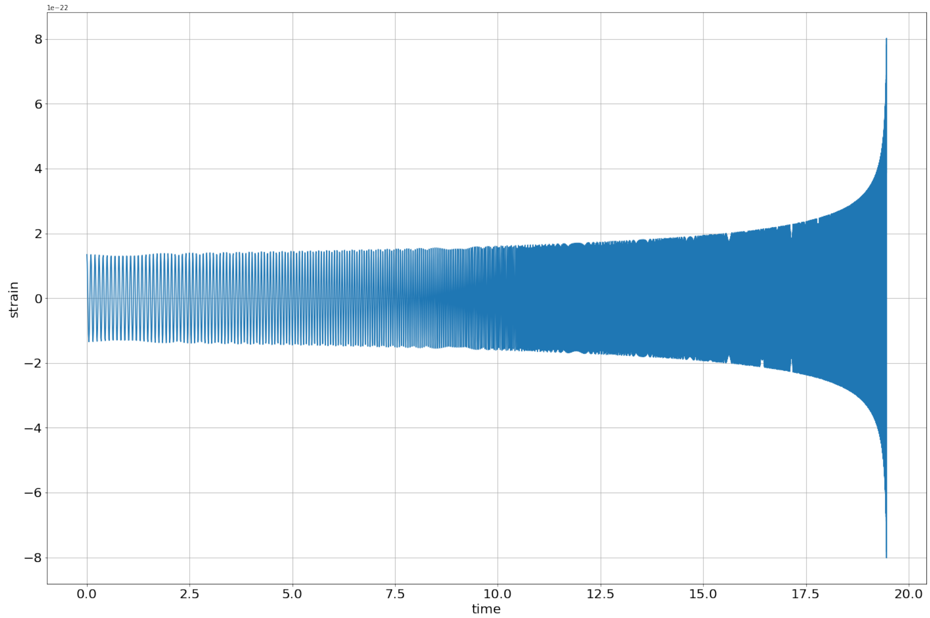
(c) Masses $25M_{\odot}$ and $35M_{\odot}$



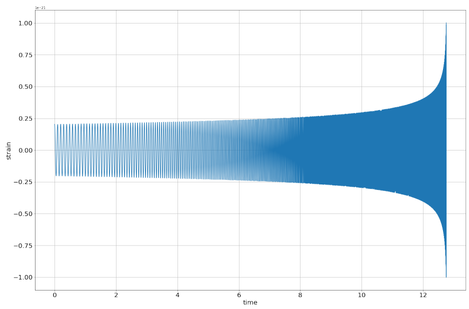
(d) Masses $30M_{\odot}$ and $35M_{\odot}$

Figure 3.2: Binary system's gravitational waves waveform obtained using the Newtonian approximation

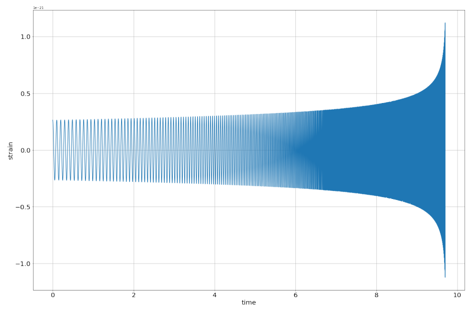
3.2 Post-Newtonian Corrections



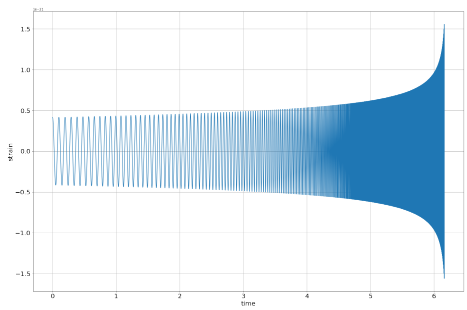
(a) Binary system's (with masses $15M_{\odot}$ and $15M_{\odot}$) gravitational waves waveform obtained using the PN-expansion



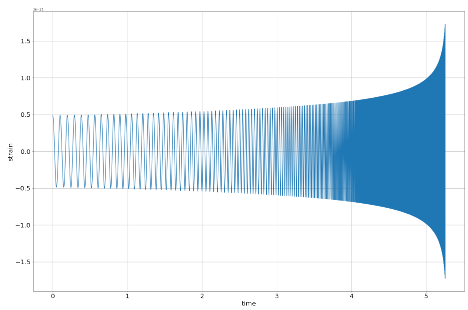
(b) Masses $15M_{\odot}$ and $25M_{\odot}$



(c) Masses $15M_{\odot}$ and $35M_{\odot}$



(d) Masses $25M_{\odot}$ and $35M_{\odot}$



(e) Masses $30M_{\odot}$ and $35M_{\odot}$

Figure 3.3: Binary system's gravitational waves waveform obtained using the PN-expansion

3.3 Comparison of time of coalescence

The post-Newtonian corrections lead to a delayed coalescence time as compared to the Newtonian approximation. In other words, the post-Newtonian model has the frequency rise slightly slower than the Newtonian model.

We estimate time of coalescence t_c by the time for any particular model at which the frequency reaches f_{ISCO} (1.38), the frequency of the Innermost Stable Circular Orbit (ISCO), and call this estimate t_{ISCO}.

This delay varies with the mass, which is discussed in 3.3.1.

Below are some plots showcasing this for both black holes having mass $35M_{\odot}$.

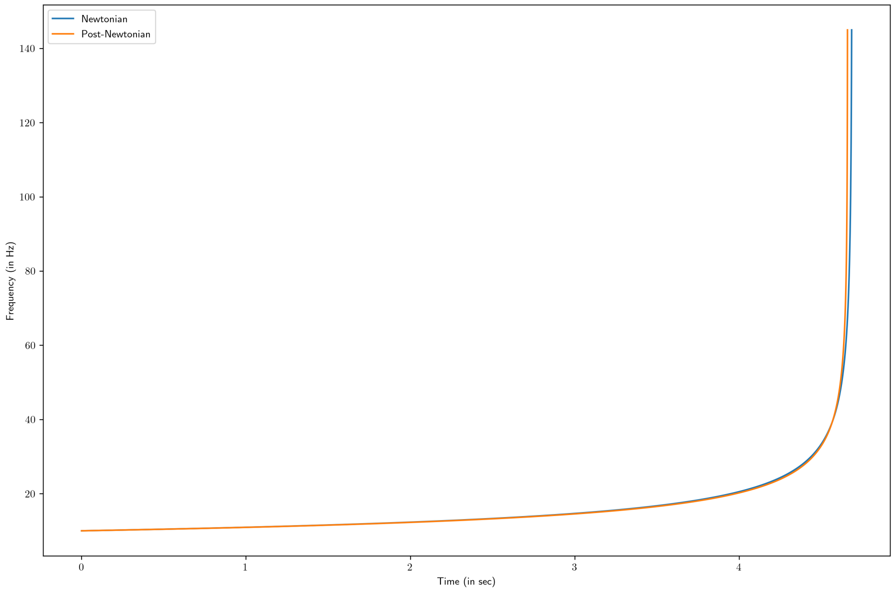


Figure 3.4: The variation of frequency with time for the Newtonian and Post-Newtonian models, starting from $f^* = 10$ Hz ,$t^* = 0$ sec (with both masses $35M_{\odot}$)

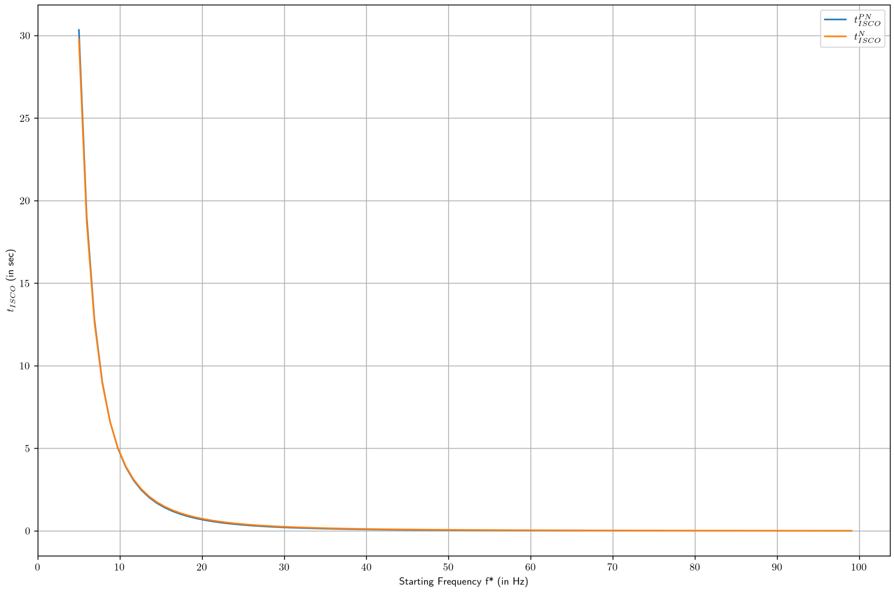


Figure 3.5: The variation of t^N_{ISCO} and t^{PN}_{ISCO} with the initial frequency (with masses $35M_\odot$ and $35M_\odot$)

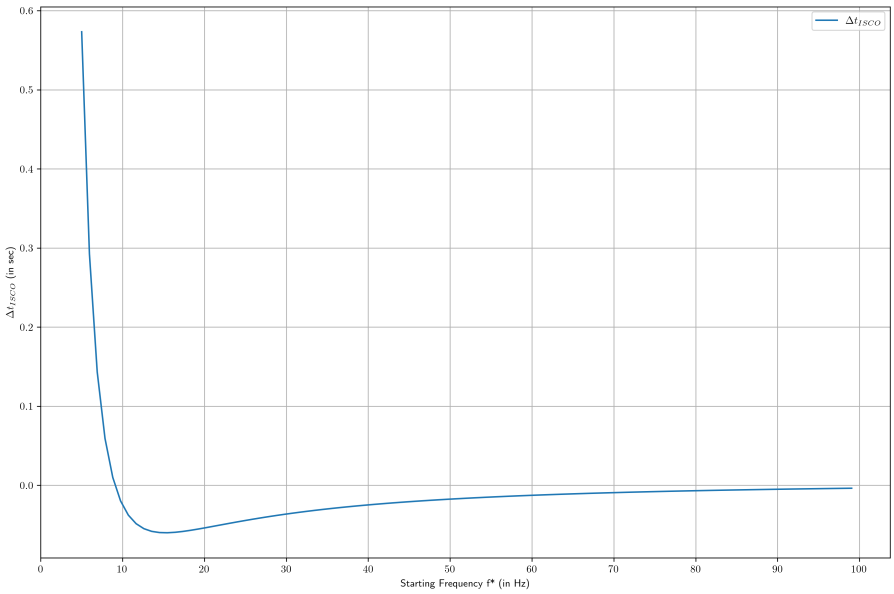
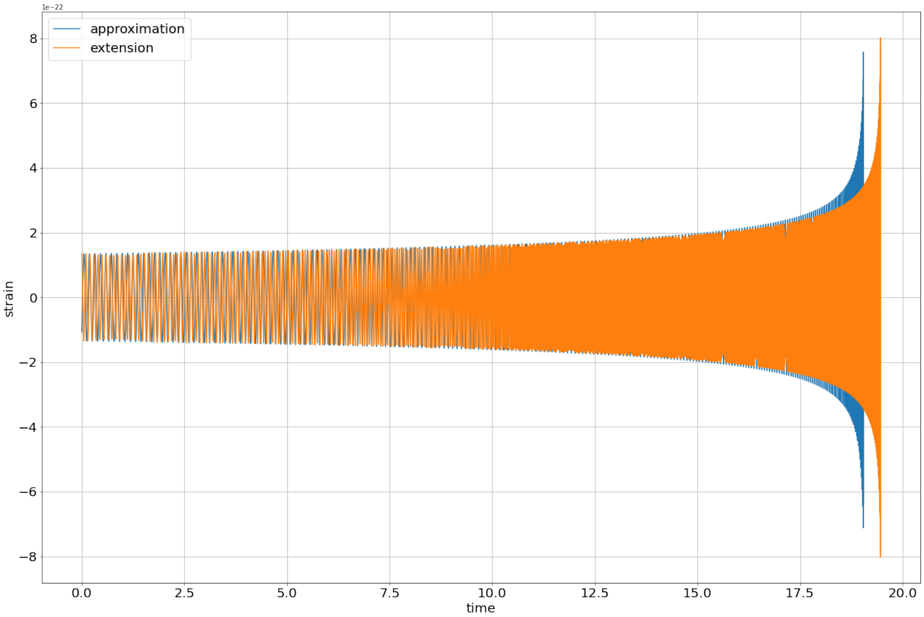
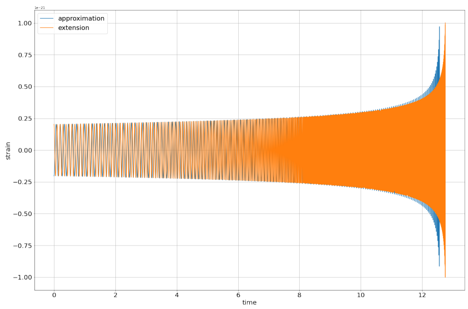


Figure 3.6: The variation of $\Delta t_{ISCO} = t^{PN}_{ISCO} - t^{N}_{ISCO}$ with the initial frequency (with masses $35M_\odot$ and $35M_\odot$)

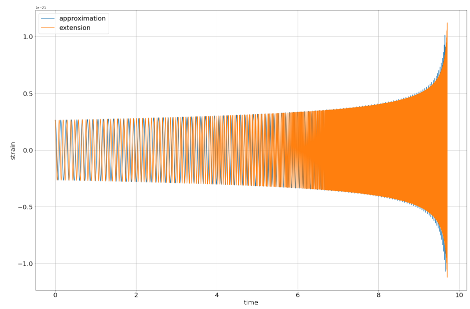
Below, the complete wave-forms for both models (N and PN) have been plotted in strain vs time graphs for various masses.



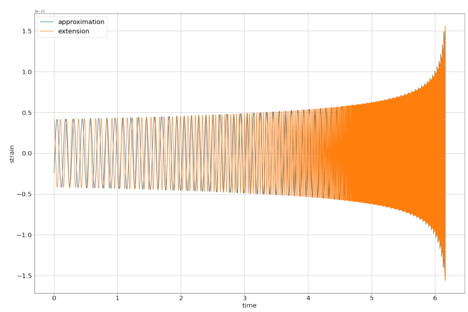
(a) Binary system's (with masses $15M_{\odot}$ and $15M_{\odot}$) comparison of gravitational waves waveforms gained via PN-expansion and Newtonian approximation



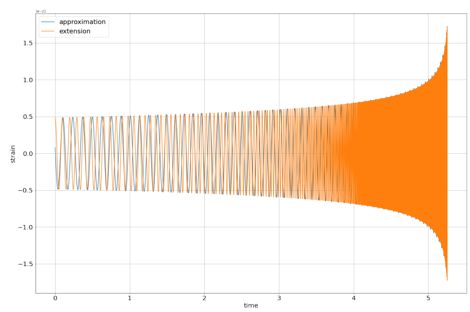
(b) Masses $15M_{\odot}$ and $25M_{\odot}$



(c) Masses $15M_{\odot}$ and $35M_{\odot}$



(d) Masses $25M_{\odot}$ and $35M_{\odot}$



(e) Masses $30M_{\odot}$ and $35M_{\odot}$

Figure 3.7: Binary system's comparison of gravitational waves waveforms gained via PN-expansion and Newtonian approximation

3.3.1 Variation of delay with Mass

In an equal mass binary system with the aforementioned PN corrections the delay in coalescence time compared with Newtonian approximation results show following trend. The mass corresponds to the member of a binary system.

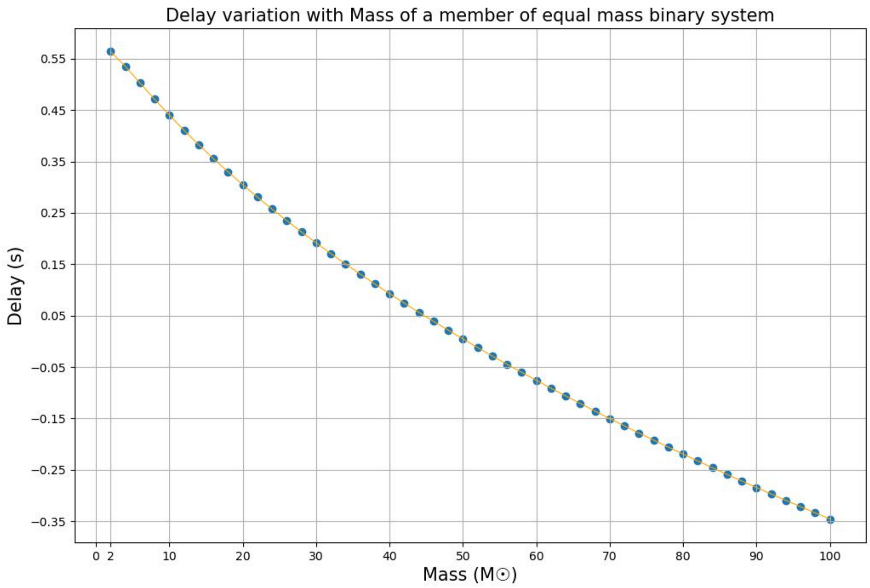


Figure 3.8: The variation of delay in time of coalescence with the mass of a member of an equal mass binary system.

The Delay decreases with the Mass. The value for delay stays positive for mass values below $50M_{\odot}$ and continues decreasing further, which corresponds to shorter time of coalescence for systems with Post Newtonian corrections compared to Newtonian Approximation.

This decreasing trend in delay with increasing mass can be partly explained due to the limits within which Post Newtonian Theory is effective. It was discussed earlier that the Post Newtonian Formalism is applicable only for weak self-gravitating sources. Beyond the threshold value of $50M_{\odot}$, the gravitational fields of our binary system are no longer weak and cannot be regarded as moderately relativistic. Therefore, we start getting a negative delay or difference in the time of coalescence for the two waveforms. Another reason behind this negative delay could be related to the radius of Innermost Stable Circular Orbit (R_{ISCO}) and frequency at ISCO (f_{ISCO}). As masses of the members of binary system increases the corresponding Schwarzschild radii will increase and the value of R_{ISCO} will also increase. Consequently, the black holes in our binary system would reach the merger phase earlier as compared to the lower mass counterparts, resulting in a shorter length of the inspiral phase of waveform and lower f_{ISCO}. The Post Newtonian correction terms may exaggerate this and have an opposite effect on the length of the waveform due to lower frequency values. As a result, the Post Newtonian waveform has a shorter length than the Newtonian approximation waveform giving rise to a negative delay for higher masses.

4. Matched Filtering

4.1 Concept of Matched Filtering

The strain data collected by LIGO and VIRGO interferometers is prone to external noise which is substantial enough to obscure any gravitational wave signal. This is because we are attempting to detect changes in space of the order of almost $10^{-20}m$. At such exceedingly small scales, even the slightest of disturbances due to seismic activities, instrumental noise, etc. can induce unwanted noise in the data. Although, numerous statistical methods can be used to reduce the noise, a firsthand methodology is still necessary. Matched Filtering is one such method that can be used to detect the presence of gravitational waves that are loud enough in a noisy data. The strain data collected by LIGO and VIRGO detectors is a time-series that can be decomposed into a noise component and a signal component (if there is any):

$$h(t_i) = n(t_i) + s(t_i) \tag{4.1}$$

It is usually assumed that the noise component is static in time and follows a Normal (Gaussian) distribution. The process of Matched Filtering involves a bank of gravitational waveform templates which are generated using various approximants. This method relies on knowing or predicting the possible shape of gravitational wave signal in the strain data. The template waveform ($w(t)$) is then compared with the strain data ($h(t)$) by shifting the waveform against the data and calculating the cross-correlation in the time domain at each time step (assuming a static and Gaussian noise allows us to do so). This cross correlation output is also defined as the Signal to Noise Ratio or the SNR times-series ($\rho(t)$).

$$\rho(t) = 2\int_{-\infty}^{\infty} h(t') \cdot w(t'-t)dt' \tag{4.2}$$

Whenever the template waveform matches with the gravitational wave signal in strain data we get a spike in the Signal to Noise Ratio. This spike in SNR is also called as a "trigger". Thus, a higher value of SNR localized at a point in time can be interpreted as presence of a gravitational wave signal in the noisy data. The process of matched filtering is carried over for different template waveforms in the template bank and the template with highest SNR is chosen to get an estimate of source parameters. The portions of data with high enough SNR is often taken for a Chi-squared (χ^2) test of hypothesis to eliminate any possibility of instrumental glitches giving rise to false triggers.

4.2 Matched Filtering of GW150914

4.2.1 Retrieving and Pre-conditioning the strain data for GW150914

We chose the GW150914 merger event for Matched Filtering with the generated waveforms. GW150914 was the first ever direct observation of gravitational waves in 2015 that originated from the merger of two black holes at a distance of around 500 mega-parsecs from Earth. The two black holes had a nearly equal mass of about 30 solar masses. In the generated waveforms, we assumed a mass of 35 solar masses for both black holes. The strain data for GW150914 event is available on the LIGO GWOSC website and can be retrieved manually or through Python interface and packages like *GWOSC, GWpy* or *PyCBC*. The time-series of 32 second strain data of GW150914 was retrieved using *PyCBC*:

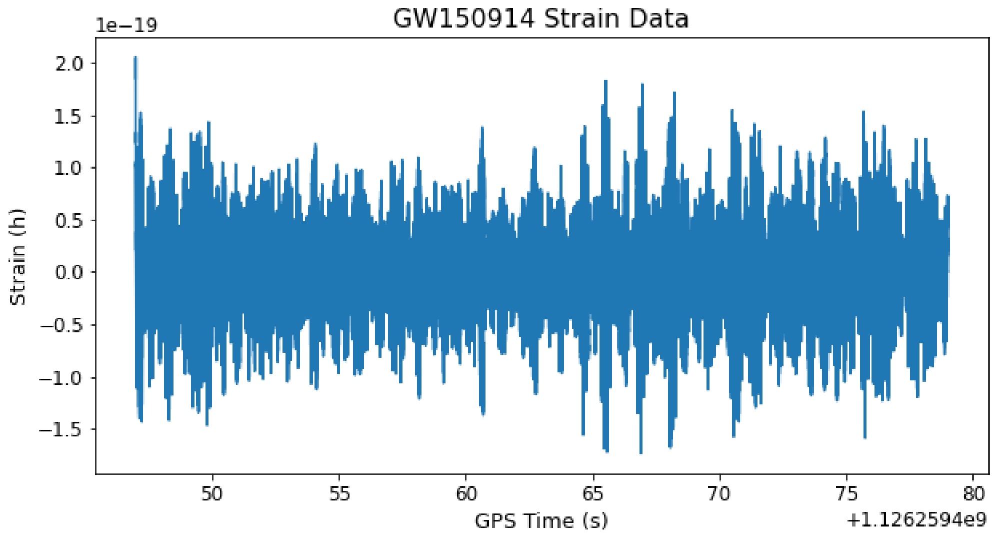


Figure 4.1: LIGO Strain Data for GW150914

The data has a sampling frequency of 4096 Hz, which gives a sampling time of around 0.00244498777 seconds. Here, the time along X-axis is the GPS time measured in seconds from the start of the GPS Epoch.
This data needs to be pre-conditioned to ensure optimal matched filtering. We first bandpassed the data in the frequency range of 10-512 Hz. This range corresponds to the frequency band in which the LIGO detectors are most sensitive. This eliminates the low and high frequency noise components from our data. Next, we trim our data length to match the length of generated waveforms as this is essential in carrying out the cross-correlation.

4.2.2 Matched Filtering with the Newtonian approximation Waveform

The plot of strain data in Figure 4.1 is filled with an overwhelming amount of noise. Our gravitational wave signal is buried somewhere in this noise. First we carry out Matched Filtering of this data with the approximate Newtonian waveform generated in the previous section. The following plot of SNR timeseries is obtained:

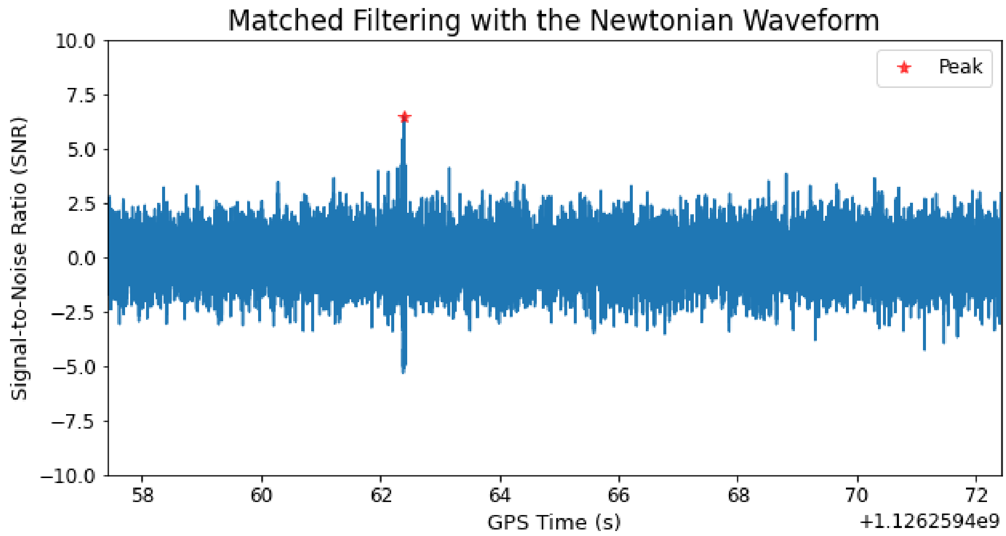


Figure 4.2: Matched Filtering with Newtonian Waveform

A discernible spike in the SNR is observed at around 1126259462.394287 seconds GPS time. The matched filtering process returns negative values of SNR. This is due to a complex valued SNR associated with the template component that is out of phase with the data by $\pi/2$. Since the phase of signal can be anything, we take absolute values of the SNR and plot them:

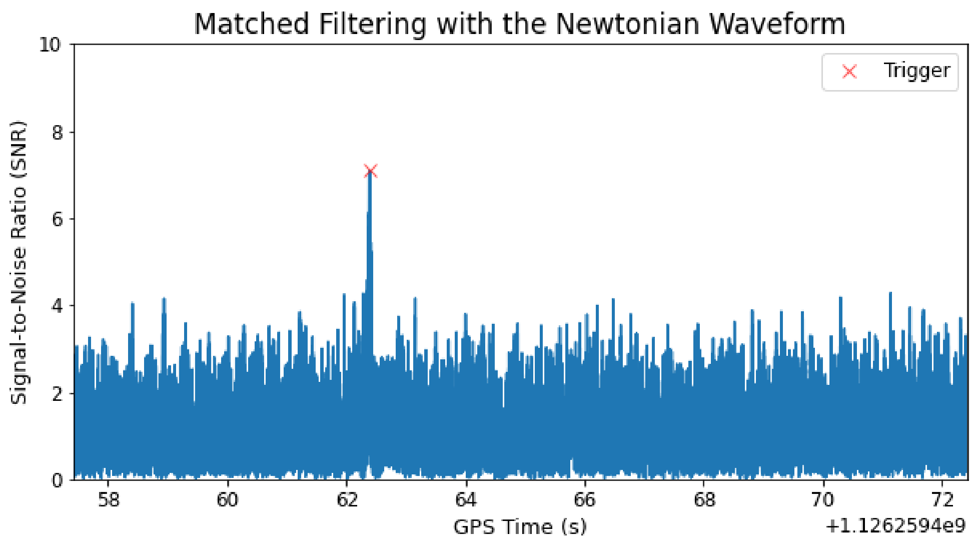


Figure 4.3: Maximized Signal to Noise Ratio Plot

We get a peak SNR value of about **7.0931** at 1126259462.3891602 seconds GPS time. Hence, the Newtonian waveform yielded a sufficiently good Signal to Noise Ratio with Matched Filtering. Further processing of the strain data like whitening can give a better SNR value. The GPS time at which the peak is observed matches with the GPS time of 1126259462.4 seconds as mentioned in the GWOSC Event Catalog, to a reasonable degree of accuracy.

4.2.3 Matched Filtering with the PN Waveform

The Post Newtonian waveform was generated in the previous section. Matched Filtering of the data was done with this waveform and the SNR was maximized by taking absolute values as shown previously. The following SNR time series plot was obtained:

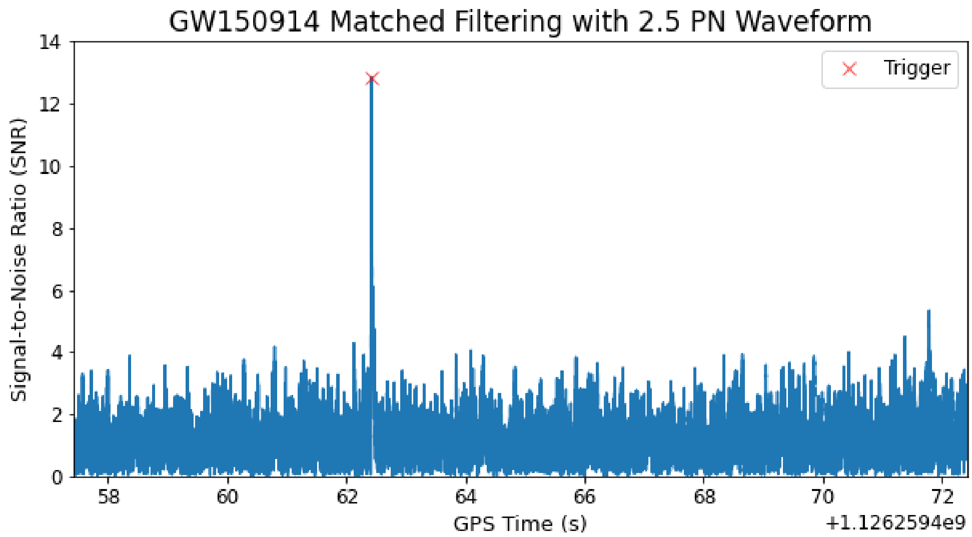


Figure 4.4: Matched Filtering with the Post Newtonian Waveform

A peak SNR of **12.83834** is observed at 1126259462.4233398 seconds GPS time. This value is nearly twice the value of maximum SNR obtained using the approximate Newtonian waveform.

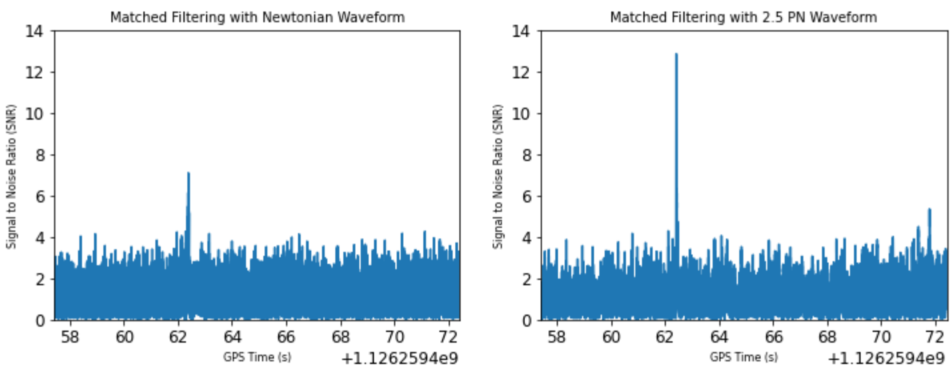


Figure 4.5: A side by side comparison makes this difference more apparent.

Furthermore, the time at which the trigger is observed (1126259462.42) matches with the event GPS (1126259462.4) according to the GWOSC Event Catalog. It is evident that the Post Newtonian waveform is more efficient in finding gravitational wave signals buried in the data than just the Newtonian waveform. This is because the frequency as well as the phase evolution terms in the waveform receive Post Newtonian corrections which introduce a delay in the time of coalescence as compared to the Newtonian waveform. The delay causes our signal (which we assume to be from a Post Newtonian source) to remain in the detector for a longer time hence resulting in a higher Signal to Noise Ratio. Consequently, it can be shown that PN corrections of higher orders will give an even better SNR.

III

Conclusion

5. Conclusion and Discussion

This project was an intermediate approach to the formalism of General Relativity and gravitational wave signal modeling. While, many accurate models like Numerical Relativity and Effective One-Body Simulation exist, the Newtonian and Post Newtonian approximations can be vastly convenient and accurate in modeling gravitational wave signals for the Inspiral phase of waveform, even till and at the merger. The concreteness of Newtonian Gravity is proved, when by just adding a simple time delay to the Newtonian Potential explained a seemingly relativistic phenomenon. In Post Newtonian theory, we still partly adhered to the Newtonian Theory whilst adding the PN Corrections. The effectiveness of Post Newtonian Expansions was evident from our results of Matched Filtering. Naturally, a computation of higher PN order would yield much better results by considering effects like Radiation Reaction and the "Hereditary terms". However, several difficulties arise when using the Post Newtonian Expansions at higher orders. These difficulties occur mostly in the form of divergent integrals when either of the source masses go beyond a certain threshold value. Furthermore, the Post Newtonian formalism breaks down for boundary conditions at infinity ($r \to \infty$).

Nevertheless, the Post Newtonian Theory continues to serve as a powerful tool in studying non-relativistic to moderately-relativistic phenomenon. With the advent of new detectors and technologies like LIGO India, we are bound to discover many more merger events and gravitational wave signals where these models will prove valuable as a first step in the process of data analysis, signal modeling and parameter estimation.

IV Bibliography

6. References

- Sean M Carroll. Lecture notes on general relativity. arXiv preprint gr-qc/9712019, 1997.
- Michele Maggiore. Gravitational waves: Volume 1: Theory and experiments. OUP Oxford, 2007.
- Bernard F Schutz. Gravitational waves on the back of an envelope. American Journal of Physics, 52(5):412–419, 1984.
- LIGO Scientific, VIRGO collaborations, BP Abbott, R Abbott, TD Abbott, MR Aber-nathy, F Acernese, K Ackley, C Adams, T Adams, P Addesso, et al. The basic physics of the binary black hole merger GW150914. Annalen der Physik, 529(1-2):1600209, 2017.

The Python Codes from the project are avaliable at GITHUB/Arhaan/KSP-Gravitational-Waves